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II. *The Cochlea of the* Ornithorhynchus platypus *compared with that of ordinary Mammals and of Birds.*

By URBAN PRITCHARD, *M.D., F.R.C.S., Aural Surgeon to King's College Hospital.*

Communicated by Professor HUXLEY, *Sec. R.S.*

Received November 9,—Read December 9, 1880

[PLATES 45, 46.]

IN the course of my researches into the structure of the cochleæ of various Mammals and Birds, I have been frequently struck with the marked dissimilarities displayed in these two forms of the organ. For this reason I naturally became anxious to examine it in a Monotreme, and to find, if possible, some links in structure between these two cochleæ.

I therefore made inquiries for specimens of the *Ornithorhynchus platypus* among my Australian friends, and, after a year or two's patience, succeeded in obtaining several skulls of that animal in an excellent state of preservation. These have enabled me to make some researches into the morphological and microscopical structure of this cochlea, the results of which I trust will be deemed worthy of the consideration of the Royal Society.

Morphology.

That part of the bony labyrinth which constitutes the cochlea of this animal consists of a somewhat curved tube, about $\frac{1}{4}$ inch (6·3 millims.) in length and $\frac{1}{20}$ inch (1·26 millim.) in diameter, projecting from the cavity of the vestibule and embedded in the substance of the petrous bone near its anterior extremity (see Plate 45, fig. 1).

This tube is directed almost horizontally forwards and is slightly curved outwards, its apex forming a little projection on the outer border of the petrous bone close to its pointed end.

In section this tube is of an oblong form at its commencement, vestibular end, with its greatest measurement ($\frac{1}{13}$ inch or 2 millims.) from top to bottom (see Plate 45, fig. 3). A little further forwards it becomes triangular, but soon it assumes its general oval-shape with its greatest diameter ($\frac{1}{22}$ inch or 1·15 millim.) from side to side (see Plate 45, fig. 2). The tube terminates in a slightly enlarged rounded extremity, which, however, is not spherical but flattened from top to bottom. Its greatest diameter (from side to side) is $\frac{1}{15}$ inch or 1·7 millim. This enlargement is due to the partial coiling up of the anterior extremity of the membranous labyrinth, showing the tendency to assume the spiral form: thus, the ductus cochleæ of the *Ornithorhynchus* may be likened to that of a typical Mammal which has been unwound except at its extreme apex.

The general form of the Duckbill's cochlea contrasted with that of the typical Mammal[*] *and Bird.*[†]

On comparing the general form of these three cochleæ it is at once evident that there is much more resemblance between those of the Duckbill and Bird than between either of these and that of the typical Mammal. Thus the cochleæ of both Duckbill and Bird consist of curved tubes enlarged at their anterior extremities, with not much difference in diameter throughout except at their apex.

These two differ, however, in that the Duckbill's is more curved; that it is curved *outwards*, whereas the Bird's is curved *inwards* (towards the median line of the body); and in that the enlargement at the apex in the latter is more oval in shape instead of being rounded.

The differences between the general form of this Monotreme's cochlea and that of other Mammals will be found to be very marked. Whereas the former is merely a curved tube of nearly equal diameter throughout, the latter is made up of a coiled tube which tapers from commencement to apex so as to assume the form of the inside of a Snail's shell.

The length of the tube is much greater in typical cochleæ; thus in the smaller Mammals, such as the Cat, Dog, and Rabbit, it measures about $\frac{3}{4}$ inch (19 millims.) against $\frac{1}{4}$ inch (6·3 millims.) in the Duckbill. Lastly, the axis of the spiral cochlea is horizontal, whereas that of the curved tube is vertical.

The internal arrangement and minute structure of the Duckbill's cochlea.

It may be well to premise that on examining the interior of this cochlea the resemblance between it and that of the Bird is almost entirely lost, and its true Mammalian type becomes at once apparent.

The interior of the tube is divided horizontally into two unequal parts by means of a lamina, the upper division being larger than the lower. The outer and larger portion of this horizontal lamina is bony, the inner portion membranous. A very delicate membrane proceeds from the upper surface of the osseous lamina, near to its inner margin, to the inner wall of the cochlea tube; this membrane forms an acute angle with the inner portion of the horizontal lamina, and thus cuts off a third tube, which is triangular in section and very much smaller than the other two divisions of the cochlea tube.

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[*] For description of Mammalian cochlea see QUAIN's 'Elements of Anatomy,' 8th ed., vol. ii., and STRICKER's 'Histology,' vol. iii.

[†] For description of the Bird's cochlea see—

PAUL MEYER, 'Labyrinthe Membraneux du Limaçon chez les Reptiles et les Oiseaux.'

HASSE, Die Schnecke der Vogel in v. Siebold's und Kölliker's Archiv. für Wissensch. Zoologie, Bd. xvii., p. 381, 1867.

HASSE, Nachträge zur Anat. der Vogelschnecke, ibid., p. 461.

DEITERS, Untersuch. u. d. Schnecke d. Vögel Archiv. f. Anat. u. Phys., p. 409, 1860.

The upper large tube is the scala vestibuli, and communicates posteriorly with the general cavity of the vestibule; the small triangular tube is the scala media or ductus cochleæ, and communicates with the saccule of the vestibule. The lower tube is the scala tympani, and communicates posteriorly with the tympanic cavity in the dried bone.

The scala vestibuli is largest at its vestibular end; here it is nearly square, being $\frac{1}{24}$ inch or 1·05 millim. high, by $\frac{1}{25}$ inch or 1 millim. broad. Further forwards it becomes smaller and alters its form until it becomes oval, with its greatest diameter $\frac{1}{22}$ inch or 1·15 millim. from side to side, and $\frac{1}{43}$ inch or ·58 millim. from top to bottom. Near the apex of the cochlea the scala vestibuli communicates with the scala tympani by means of an oval opening--helicotrema—formed by the absence of the bony lamina, which does not extend as far forward as the apex of the cochlea (see Plate 45, fig. 1).

The scala tympani is throughout much smaller than the scala vestibuli, and this is due to the fact that the osseous lamina encroaches much further inwards on its lower than on its upper side. The scala tympani is nearly circular in section, and varies very slightly in diameter, $\frac{1}{45}$ inch or ·56 millim. throughout its entire length. At its anterior extremity it unites with the scala vestibuli, as already stated.

These two scalæ are lined with endothelium and contain epilymph. In one of my specimens the scala tympani is almost completely filled, especially its posterior portion, with a substance resembling organising lymph, the true nature of which I am at present unable to explain.

Before passing on to the description of the scala media it will be more convenient to describe the lamina ossea, or outer* portion of the horizontal lamina which divides the general tube. This corresponds both in position and structure to the lamina spiralis ossea of the typical Mammal.

It is a wedge-shaped mass attached by a very broad base to the outer wall and floor of the cochlea tube, thus encroaching much more on the lower than the upper half of this tube, as already stated.

The upper surface of the lamina, except near its attachment to the wall, is almost horizontal; but just before its free margin it is raised up into a rounded eminence—the limbus—and this terminates in a sharp edge the upper lip of the marginal sulcus.

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* In describing the position of the parts in the spiral cochlea it is usual to consider the modiolus or axis as the centre; but in describing the curved cochlea of the Duckbill the median line of the body is taken as the centre As a result of this difference, the relative meaning of inner and outer is reversed; thus the outer part of the spiral cochlea corresponds to the inner of the Duckbill's.

The lower surface of the lamina slopes upwards and inwards from the floor of the tube, curving round so as to become almost parallel to the upper surface. It also terminates in a sharp edge the lower lip of the marginal sulcus; this lower lip projects much further inwards than the upper.

The free margin of the lamina is deeply grooved, forming the sulcus just alluded to.

The lamina ossea does not extend to the apex of the cochlea, but terminates beforehand in a curved border, which forms part of the boundary of the oval communication between the scala vestibuli and tympani. As the lamina approaches its anterior termination it gradually loses its wedge-shape and ceases to be attached to the outer wall of the tube; so that at last it simply consists of a plate springing from the floor and curving round so as to become horizontal at the limbus.

The substance of this lamina ossea is modified bone, like the lamina spiralis ossea of typical Mammals. It is not so dense as ordinary bone, and chiefly consists of what may be called ossified stellate fibrous tissue. It is well supplied with blood vessels, which have a tendency to become tortuous. The substance of the limbus is again modified, being more homogeneous in structure and not unlike the matrix of hyaline cartilage; moreover, its surface is deeply grooved transversely, giving it a dentate appearance, the free edges of the teeth forming the sharp upper lip of the marginal sulcus.

Through the lamina, close to its lower surface, run the bundles of nerve fibres on their way to the organ of CORTI; these will be described further on.

From this brief description it will be seen that the lamina ossea agrees entirely in structure and position with the lamina spiralis ossea of the ordinary Mammal.

The Scala Media or Ductus Cochleæ.—This constitutes the membranous labyrinth of the cochlea. Taken as a whole it is a triangular tube rather more than $\frac{1}{4}$ inch in length which is coiled round at the apex of the cochlea, and is connected posteriorly with the saccule of the vestibule by means of a circular tube with very delicate membranous walls, in a manner that I have not completely made out. As already mentioned, it is triangular in section. The floor or base of the triangle is formed by the limbus of the osseous lamina and by the membrana basilaris, which stretches from the lower lip of the marginal sulcus to the ligament of the cochlea. This ligament is a large cushion of connective tissue adherent to the inner wall of the cochlea, and forms the inner boundary of the scala media. The third and upper side of the triangle is formed by the delicate membrane of REISSNER, which stretches from the upper part of the ligament to the upper surface of the osseous lamina, being attached just where the elevation of the limbus commences.

The whole of the inner surface of the scala is lined with epithelium, which is vari ously modified according to its position,—that part of this epithelium which lies on the central portion of the floor being developed into that complex form of nerve epithelium known as the organ of CORTI.

The membrana basilaris is a strong membrane made up of three layers. The lower consists of fibres running lengthways, in the direction of the scala; these fibres have spindle-shaped nuclei, which present a rounded appearance like the nuclei of epithelium when seen in transverse sections of the cochlea. The next layer, the thickest, is more homogeneous in appearance, its component fibres run transversely from the lower lip of the marginal sulcus to the ligamentum cochleæ with which they blend. The upper layer is simply composed of one layer of fine fibres covering the inner part of the second layer; these are continuous with the rods of CORTI, to be described later.

The ligamentum cochleæ is a large cushion of connective tissue, somewhat triangular in section; its inner surface is closely adherent to the wall of the cochlea, and its outer one is raised up into two horizontal ridges one above the other. The lower is pointed and situated opposite the lower lip of the sulcus; to this ridge the membrana basilaris is attached: the upper is rounded off, corresponds and is opposite to the upper lip of the sulcus. The grove between may be called the secondary sulcus.

The connective tissue of the ligament is made up chiefly of fine interlacing fibres, but those composing the lower ridge are much coarser and transparent, they radiate from the point of attachment of the membrana basilaris. The whole substance of the ligament is very vascular.

The membrane of REISSNER is composed of a very delicate basement membrane, which is exceedingly difficult to demonstrate, and a layer of cells on each side—those on the outer or upper side being the endothelial cells of the scala vestibuli, and those on the inner or under side the epithelial cells lining the scala media. These epithelial cells are hexagonal in form and arranged in a single layer; they are much thicker than the endothelial cells, and have larger nuclei.

Here and there on the upper surface of the membrane of REISSNER are found blood-vessels, which pass across from the lamina to the ligament, and in certain spots a blood-vessel becomes convoluted so as to form an elevated knot (see Plate 46, fig. 7).

The epithelium lining the Scala Media.—That portion which lines the membrane of REISSNER has just been described, and if the cells of this layer be traced downwards on to the limbus they will be seen to become more rounded in form. At first they cover the surface of the limbus, but presently they are seen dipping down into the grooves between the teeth of the limbus, getting deeper and deeper, and thus entering the marginal sulcus without covering the teeth at all. The cells lining the sulcus and outer part of its lower lip are still rounded, but before arriving at the point of the lower lip the cells become entirely altered so as to form the organ of CORTI, which will be described later. On the other side of the organ of CORTI the epithelium is cuboid, and in this form it lines the inner part of the membrana basilaris and the surface of the ligamentum cochleæ to a little beyond the upper ridge. From this point to the attachment of the membrane of REISSNER the epithelium is peculiarly modified, and is traversed by numerous blood-vessels which chiefly run longitudinally (to the cochlea); this is the stria vascularis.

The shape of these epithelial cells is irregular, somewhat resembling transitional epithelium but packed closer together. Numerous blood-vessels traverse the layer but never appear quite on the free surface. As the form of epithelium, which is modified so as to receive the terminations of special nerves, is called "nerve epithelium," that of the stria vascularis might, I think, be very appropriately called "vascular epithelium."

The epithelial lining has now been traced back to the membrane of REISSNER, and thus right round the tube, with the exception of the organ of CORTI, which has now to be described.

As the organ of CORTI of the *Ornithorynchus* (see Plate 46, fig. 6) closely resembles that of the ordinary Mammal, which I described in a paper read before this Society in 1876,[*] I propose to give merely a brief sketch of its structure, and later on to note how it differs from the typical Mammal.

The organ of CORTI consists of two rows of rods separated from each other below, and united at their upper extremities, thus forming with the membrana basilaris, on which they stand, a triangular tunnel.

Each of the rods have enlarged upper and lower extremities; the latter are similar in both rods, being simply enlarged feet. But the form of the upper extremity differs in the two rows; in the rods of the outer one it is oblong, with a delicate process projecting inwards from its upper part. The upper extremity of the inner rods is more rounded and fits into a shallow concavity of the head of the outer rod. From the upper part of these inner rods there is also a delicate process projecting inwards and lying below that of the outer rod. The shafts of both rows of rods are cylindrical and of equal size.

On each side of the rods are arranged rows of cells, there being a greater number on the inner side (the side next to the ligamentum cochleæ). The most important of these are the hair or bristle cells which are arranged in four rows;[†] one row to the outer side of the outer rods, and three rows to the inner side of the inner rods. These bristle cells are elongated with flattened upper extremities and rounded lower ones; their outline is well defined, and they contain a well-marked rounded nucleus; from their summit projects a row of four or five bristles.

[*] 'Proc. Roy. Soc.,' vol. xxiv., p 346. See also—

U. P., "Structure and Function of the Rods of the Cochlea," 'Monthly Micros. Journ.,' 1873.

U. P., "Development of the Organ of Corti," 'Journ. of Anat. and Phys.,' vol. xiii.

WALDEYER, "The Cochlea," STRICKER'S 'Histology,' vol iii.

LAVDOWSKY, "Untersuch. u. d Akustichen Endapparat. d. Säugeth.," 'Archiv. Micros. Anat.,' p. 497, Bd. xiii, 1877.

BÖTTCHER, "Ueber Entwick. u. Bau des Gehorlaby. u. Untersuch. a. Saugeth.," Nova Acta, Leopold. Acad, Bd. xxxv.

GOTTSTEIN, "U. d. fein. Bau u. d. Entwick. d. Gehörschnecke d. Säugeth.," 'Archiv. f. Micros. Anat.,' Bd. viii., p. 145.

WINIWARTER, "Untersuch. u. d. Gehörschn. d. Säugeth.," 'Wien. Akad. Sitzber.,' Bd. lxi., 1 Abth., p 683, 1870.

[†] See postscript for the account of an additional row.

A reticulate membrane covers and is closely adherent to the upper surface of the rod heads, and extends outwards and inwards over the summits of the bristle cells—the bristles passing through rounded meshes of the membrane. This membrana reticularis is made up of one row of circular meshes (annulæ) to the outer side of the rods, and on the inner side three alternating rows of annulæ with elongated meshes (phalanges) between.

Passing downwards from the inner portion of the membrana reticularis to the membrana basilaris we find three or four rows of fibres (trabeculæ), which run almost parallel to the inner rods, and are attached to the latter membrane by enlarged feet very like those of the rods. Between these trabeculæ and below the level of the bristle cells are three or four rows of nuclear cells—that is to say, cells with well-marked nuclei, but apparently no cell-walls; these are the cells of DEITERS.

In each of the lower angles of the triangular tunnel is situated a row of nuclear cells similar to the cells of DEITERS, and to the outer side of the outer rods are a number of these nuclear cells lying on the under lip of the sulcus.

The inner and outer boundary of the organ is formed of irregular columnar cells (the supporting cells), which are the transitional link between the ordinary epithelial layer and the nerve epithelium proper.

The nerve filaments which come through a row of holes on the upper surface of the lower lip of the sulcus (habenula perforata), pass into the organ of CORTI, and no doubt terminate in the nuclear cells and bristle cells; but this point I have not been able to demonstrate in the *Ornithorhynchus.*

To complete the description of the ductus cochleæ there only remain its commencement and termination to be considered. The former I have not yet been able to trace out, but with the latter I have been more fortunate.

The organ of CORTI and the rest of the lamina membranacea is not continued further forwards than the end of the lamina ossea; but the ductus does not terminate here, for it is continued round the circumference of the spheroidal extremity of the bony tube, and ends in an enlarged cul-de-sac close to the base of the lamina ossea. Thus the ductus makes a turn of about three-quarters of a complete circle at the apex of the cochlea tube. This will be better understood by referring to the drawing (Plate 45, fig. 1), where it will be seen that the ductus almost completely surrounds the oval communication (helicotrema) between the scalæ tympani and vestibuli.

At the end of the lamina ossea, where the ductus commences to curve round, a complete change takes place in the shape and walls of the membranous labyrinth. It first becomes contracted into a circular tube, and after passing the extreme apex of the cochlea it enlarges again into an ovoid chamber—the lagena (see Plates 45, 46, figs. 4, 5). The walls of this chamber are formed by a thick layer of dense connective tissue, lined internally by a single layer of epithelium. The cells of this epithelial layer are for the most part cuboid, but near the further end of the lagena a patch of nerve epithelium is found; this patch extends round the chamber, but is broader on its inner wall.

This nerve epithelium is very similar to that of the maculæ acusticæ of the saccule and utricle*; thus it consists of a deep layer of small rounded cells connected with a superficial layer of ciliated ones—thorn cells and bristle cells. These are covered by a perforated cuticular membrane, through which the bristles and thorns pass; the whole is surmounted by a loose mucoid mass, into which the thorns and bristles project, and which no doubt contain otoliths in the fresh state.†

The last bundles of the cochlear nerve pass forwards to supply this acoustic spot, and terminate in the nerve epithelium in the same manner as in the saccule and utricle of the vestibule.

The course of the cochlear branch of the auditory nerve.—This passes through the substance of the bone a little to the outer side of the cochlea tube, and just above the level of the floor. As it proceeds forwards it gives off lateral bundles of fibres to the lamina ossea, its terminal bundles going to the lagena in the manner just described.

Tracing these lateral bundles they soon enter and pass out of a ganglion, which consists of bi-polar nerve-cells, and is situated in the broad base of the osseous lamina at its commencement, but further and further inwards as it is traced forwards to its anterior end. The fibres from the ganglion collect again into bundles which divide and subdivide, uniting and reuniting so as to form a plexus; in this manner they pass through the lower portion of the lamina until the lower lip of the sulcus is reached. This is pierced with a single row of holes (habenula perforata), so that the nerve filaments may pass into the organ of CORTI. At this point the fibres lose their myeloid sheaths, the axis cylinder alone entering the nerve epithelium.

Comparison of interior of the cochlea of the Duckbill with that of the typical Mammal.

As regards the scalæ vestibuli and tympani the following differences are to be noted. In the ordinary Mammal there is not so much difference in size between the scala vestibuli and scala tympani, except at the base of the cochlea, where the latter is the larger. Again, the size of both scalæ diminishes from base to apex of the cochlea in conformity with the general diminishing spiral.

In the Duckbill the scala vestibuli is throughout larger than the scala tympani; and whereas the former diminishes its size, and alters its shape from the commencement to the end of the cochlea, the latter is throughout of the same size and shape.

The Scala Media.—The scala media or ductus cochleæ in the typical Mammal is a triangular tube, almost equilateral at its commencement at the base of the cochlea.

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* 'Quarterly Journal of Microscopical Science,' p. 397, 1876.

† In my specimens the acid decalcifying fluid has dissolved out the otoliths.

But on proceeding up the spiral, the floor becomes broader and the height lessened, so that the acute angle formed by the membrane of REISSNER and the lamina spiralis becomes more and more acute. These changes in form are not nearly so marked in the *Ornithorhynchus;* the membrana basilaris does get broader and broader, but not nearly in the same degree as in the typical Mammal. As regards the peculiar changes in the organ of CORTI, such as the lengthening of the rods, &c., which are so marked in the spiral cochlea, they are the same, but less marked in the *Ornithorhynchus* so far as I have been able to trace them out.

In the membrane of REISSNER of this Monotreme the epithelial cells are thicker than in the typical cochlea; but the great feature of difference is the presence of blood-vessels running across the membrane from lamina to ligament, forming here and there convoluted knots. These I have never found in any other Mammal, nor have they been described in this situation as far as I am aware.

The ligamentum cochleæ of the Duckbill, both as regards its shape and structure, scarcely differs at all from that of the typical Mammal; and the same may be said of the stria vascularis and its other epithelial lining.

The lamina spiralis ossea, with its limbus and sulcus, present no special points of difference. The same may be said of the membrana basilaris, with this slight exception—that whereas in the ordinary Mammal one large blood-vessel (the vas spirale) runs through its lower stratum just below the rods of CORTI, in the Duckbill two small vessels are generally to be found (see Plate 46, fig. 6)—one in the usual position, and the other a little further inwards (towards the ligament).

The organ of CORTI itself differs very slightly from that of the ordinary cochlea. There are the same number of rows of bristle cells as are found in the lower Mammals,* namely, one on the outer side of the rods and three on the inner. The trabeculæ are thicker, and their feet larger than usual; so much so, that the lower part of the first trabeculum may be easily mistaken for the foot of the inner rod.

The rods of CORTI are somewhat different; they stand more upright, so that the tunnel is not so wide; the extremities are not so well developed, the feet being smaller and the heads not so broad. I have not been able to make even an approximate calculation of the number of rods or bristle cells, but there must be a much fewer number of these in the Duckbill's cochlea than in that of the ordinary Mammal.

The course taken by the cochlear nerve and its branches differ in no essential points from those of the typical Mammal. There is in the former a ganglion very similar in relative position and component cells to the ganglion spirale. The only differences are that, whereas in the spiral cochlea the nerve trunk necessarily runs at right angles to the lamina spiralis, in this cochlea it runs parallel to the corresponding lamina; and that in the Duckbill the terminal branches pass to supply a lagena, an organ not found in the ordinary Mammal.

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* In the primates there are five and six rows of bristle cells in the upper part of the spiral, as pointed out in my paper above cited.

The last and most important difference is the presence of this lagena at the end of the membranous labyrinth (ductus) of the Monotreme's cochlea. Nothing approaching it has ever been found in the spiral Mammalian cochlea, although a very similar condition is found at the apex of the Bird's, as will be described later on. This little addition in the Duckbill forms another of the many links between the Mammal and Bird found in this peculiar animal.

A brief sketch of the Bird's cochlea[*] will now be necessary before any satisfactory comparison can be made, and it must be borne in mind that the cochlea of the Amphibia and Reptilia is similar in type, though less highly developed, to that of the Bird.

As already stated, the bony labyrinth of the Bird's cochlea is a slightly curved tube with an enlarged anterior extremity; within this is contained another more or less complete framework of dense connective tissue, cartilaginous in consistence, and very similar in structure to the so-called cartilage of the eyelid.

At the vestibular extremity of the tube this cartilaginous framework consists of two pieces, one on each side of the tube, named, from their shape, the quadrilateral and triangular cartilages (see Plate 46, fig. 8).

Stretching across the cochlea and joining these two is a strong membrane (membrana basilaris), and thus the tube is divided into two, which for convenience of description and clearness of comparison may be called the scala tympani and the ductus cochleæ.

These are not of equal size in any part of the cochlea tube; the upper or ductus cochleæ being the larger even at the commencement, and the lower or scala tympani becoming smaller and smaller until it quite disappears by ending in a cul-de-sac before the enlarged anterior extremity, the lagena, is reached.

On tracing the two cartilages forwards their triangular and quadrilateral forms become, first, more marked and then they increase in size. Not only do they enlarge generally, but their upper and lower margins extend further and further round the circumference of the tube until they meet both above and below; thus forming a complete cartilaginous tube, which is somewhat loosely adherent to the bony wall by means of connective tissue.

As this cartilaginous tube is being completed, the scala tympani is gradually encroached upon, until it is entirely filled up by the cartilage, which at length also displaces the membrana basilaris; thus there is only a single scala at the anterior end of the cochlea. Before the disappearance of the scala tympani the quadrilateral and triangular plates begin to diminish in thickness and gradually lose their characteristic forms, until in the lagena the cartilaginous tube becomes of equal thickness all round.

The roof of the ductus is membranous; and this membranous roof is not a simple arch, but is re-duplicated into a number of longitudinal (to the tube) folds of some thickness, which in many Birds fill up the greater part of the interior of the ductus

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[*] This description has been taken from my own observations, which confirm most of the statements of HASSE and MEYER.

(see Plate 46, fig. 8). This is called the tegmentum; it is a very vascular membrane, lined with large, rounded, granular, pigmented, epithelial cells, intermixed with some very transparent ones.

Tracing the epithelial layer down to the triangular cartilage on one side, and the quadrilateral cartilage on the other, the cells change their character. Those lining the upper surface of the triangular cartilage and the adjacent portion of the membrana basilaris become transitional or cuboid in shape, with well marked nuclei. Those covering the upper surface of the quadrilateral cartilage are columnar, and of these there are two kinds. The contents of those nearer to the tegmentum are peculiar, resembling in appearance the matrix of ordinary cartilage, and this is readily coloured by staining agents. Further onwards, where the surface of the cartilage curves downwards to come to the level of the membrana basilaris, these cells are replaced by other columnar cells, the contents of which are very transparent. These are again succeeded by the mass of nerve epithelium, which may be termed the organ of CORTI.

The organ of CORTI is a broad band of nerve epithelium, resting partly on the upper surface of the quadrilateral cartilage and partly on the adjacent portion of the membrana basilaris.

This organ of CORTI consists chiefly of two layers of cells. One set is composed of nuclear cells (as described in the Mammalian organ of CORTI) which lie on the membrana basilaris and surface of the quadrilateral cartilage. The other layer is situated at the surface of the organ; the cells of this layer have distinct outlines, are rounded or elongated, have distinct nuclei, and they each bear on their summit a short thorn. These thorns are formed of several short bristles united together, and they project through holes in a membrane which limits the organ, and which is very similar to the membrana reticularis of the maculæ acusticæ and serving the same purpose.

From this membrana reticularis pass down numerous fibres or trabeculæ to the membrana basilaris, binding the two membranes together.

Covering the whole organ and also some of the columnar cells of the quadrilateral cartilage is a thick mucoid layer into which the thorns project; this mucoid layer is the membrana tectoria.

The cochlear branch of the auditory nerve runs along the tube, first in the scala tympani and then in the substance of the quadrilateral cartilage. It gives off fibres all the way up, which immediately enter ganglionic cells, and then pass upwards towards that part of the upper surface of the cartilage on which the organ of CORTI lies.

Here the fibres pierce and enter the organ, not by a single row of perforations but by numerous apertures throughout the whole width of this portion of the quadrilateral cartilage. From these perforations nerve filaments pass in a radiating manner to the upper row of cells (hair cells), and possibly also to the lower cells, but their exact distribution has not been made out.

At the enlarged extremity (lagena) of the cochlea tube, which is entirely occupied by the membranous labyrinth, the lining is considerably modified, and a large patch of nerve epithelium is found, quite distinct in character and separate from the organ of CORTI. This nerve epithelium is made up of cells exactly like those found in the macula acustica of the vestibule of Mammals[*] and Birds, *i.e.*, with the alternating long bristles and thorns which project from the surface into a mucoid mass containing otoliths. The terminal branches of the cochlear nerve pass forwards to this patch of nerve epithelium, and terminate in the same manner as in the macula acustica of the vestibule.

The interior of the cochlea of the Ornithorhynchus *compared with that of the Bird.*

The differences between these two are so great that it will be well first to notice the homologous and analogous parts in each, and then to consider the similarities and dissimilarities.

In examining the drawings Nos. 2 and 8, it will be at once evident that the lamina ossea of the *Ornithorhynchus* (and other Mammals) corresponds to the quadrilateral cartilage of the Bird, and that the Mammalian ligamentum cochleæ is represented in the Bird by the triangular cartilage. Also, that whereas the lower division or scala tympani of each corresponds, the ductus cochleæ of the Bird occupies the whole of the upper division, and therefore corresponds to the scala vestibuli and scala media (or ductus cochleæ) of the Mammal. On more minute examination, the peculiar folded tegmentum of the Bird will be found to take the place of the stria vascularis and membrane of REISSNER of the Mammal; and by carrying the examination into the fœtal condition, the peculiar hyaline columnar cells of the quadrilateral cartilage will be found homologous with the teeth of the Mammalian limbus, which is represented in the early fœtus by very similar cells.

Dissimilarities in the Bird's cochlea:—

The absence of the scala vestibuli and membrane of REISSNER.

The gradual tapering and final termination in a cul-de-sac of the scala tympani.

The cartilaginous condition of the structure corresponding to the Mammalian ligament.

The cartilaginous condition of the structure which corresponds to the lamina ossea.

The cartilaginous tubular formation towards the apex of the cochlea.

The tegmentum, with its folds and peculiar pigmento-granular cells.

The absence of the marginal sulcus.

The absence of the rods of CORTI.

The large number of rows of hair cells.

The matting together of the hairs into a sort of spine.

The numerous perforations for the passage of nerve filaments into the organ of CORTI.

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[*] 'Quarterly Journal of Microscopical Society,' October, 1876.

Lastly, the ductus cochleæ is almost straight in the Bird, whereas in the typical Mammal it is spiral, and even in the *Ornithorhynchus* it is coiled round at the apex.

Similarities:—

The membrana basilaris and membrana tectoria are very much alike in both types of cochlea. Also the mode of distribution of the nerves, including the ganglionic mass.

But more important than these is the presence of a lagena with its vestibular-like nerve epithelium, which is found at the apex of the cochlea both in the *Ornithorhynchus* and the Bird, but not in that of the typical Mammal.

General résumé.

1. That although the general form of the cochlea tube of the *Ornithorhychus* is like that of the Bird (as it is almost straight and of equal size throughout, instead of being a tapering spiral), yet its internal arrangement and microscopical structure is unmistakably Mammalian in type.
2. That the acoustic apparatus of the organ of CORTI is not nearly so extensive as in typical Mammals, nor do the various minute structures forming it appear to be so well developed.
3. That the cochlea of the *Ornithorhynchus* possesses an addition (the lagena) to its membranous labyrinth which is also found in the Bird, but which is not found in the ordinary Mammal. And thus the presence of this lagena may be looked upon as another of the many links between the Bird[*] and Mammal which are found in this Monotreme.

Mode of preparation adopted in this research.

For obtaining thin sections of this cochlea I made use of the same mode of procedure as described in my paper on the organ of CORTI in Mammals, read before this Society in 1876, which is briefly as follows:—

The tissues were hardened in an alcoholic solution of chromic acid; the bone decalcified in a dilute (¾ per cent.) solution of nitric acid aided by almost constant movement of the tissue, by means of an ordinary meat roasting-jack.

Saturated with gum-water, then dehydrated in alcohol, imbedded in parafin, cut by a razor in a STIRLING's machine. Variously stained and mounted.

In order to obtain a correct idea of the relation of the parts, I mounted the sections of certain cochleæ in their proper order throughout; this I did with sections cut transversely and longitudinally, in the former case cutting the cochlea into about 120 sections, and in the latter making about 10 thicker ones. By this means, aided by unsoftened preparations filed down, I was able to get a very clear demonstration of the relative position of the parts.

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[*] But it must be borne in mind, as already stated, that the Bird's cochlea is similar to that of Reptiles and Amphibians; and therefore this lagena is really a link between the cochlea of the higher and lower vertebrates, and not merely between that of the Mammal and Bird.

But all these methods, and in fact the whole research, would have failed had it not been for the very prompt and careful manner in which the skulls were prepared and preserved before they were sent over to England, and for which kindness I desire here to offer to my Australian friends my sincere and heartfelt thanks.

Description of Plates.

Fig. 1. Longitudinal section through Duckbill's cochlea. × barely 20 diameters.

1. Opening into vestibule.
2. Apex, or anterior extremity of cochlea.
3. Helicotrema, the communication between the scalæ tympani and vestibuli.
4. Cavity of lagena; (←—≪) tube connecting it with ductus cochleæ.
5. Upper surface of lamina ossea.
6. Upper surface of organ of Corti, (α) membrane of Reissner covering organ of Corti, &c., (α′) ditto turned back.
7. Ligamentum cochleæ cut through.

Fig. 2. Transverse section through Duckbill's cochlea (middle), from No. 27 of series. × 75 diameters.

1. Scala tympani.
2. Scala vestibuli.
3. Scala media or ductus cochleæ.
4. Lamina ossea, (α) nerve with ganglion.
5. Ligamentum cochleæ, (*c*) blood-vessels, (*b*) stria.
6. Membrane of Reissner.
7. Organ of Corti, (*d*) membrana tectoria, (*e*) membrana basilaris, (*f*) marginal sulcus.

Fig. 3. Transverse section through Duckbill's cochlea (vestibular end), from No. 38 of series. × 22 diameters.

1. Scala tympani.
2. Scala vestibuli.
3. Scala media or ductus cochleæ.
4. Lamina ossea.
5. Ligamentum cochleæ.
6. Membrane of Reissner.
7. Membrana basilaris.

Fig. 4. Transverse section through Duckbill's cochlea (apex), from No. 13 of series. × 22 diameters.

1. Scala tympani.
2. Scala vestibuli.
3. Scala media or ductus cochleæ.
4. Lamina ossea, (*a*) ganglion.
5. Ligamentum cochleæ, (*b*) stria vascularis.
6. Membrane of Reissner.
7. Membrana basilaris.
8. Cavity of lagena.

Fig. 5. Transverse section through lagena of Duckbill's cochlea, from No. 13 of series. × 100 diameters.

1. Cavity of lagena.
2. Scala vestibuli.
3. Nerve epithelium.
4. Nerve fibres.
5. Mucoid mass.

Fig. 6. Transverse section of organ of Corti of Duckbill, built up from several specimens. × 400 diameters.

1. Scala tympani.
2. Scala media or ductus cochleæ.
3. Lamina ossea, (*a*) nerve.
4. Membrana basilaris, (*b*) blood-vessels, corresponding to vas spirale.
5. Triangular tunnel.
6. Marginal sulcus.
7. Outer rod of Corti.
8. Inner rod, (*c*) trabeculæ.
9. Inner hair cells.
10. Outer hair cell.
11. Cells of Deiters.
12. Extra row of hair cells.

Fig. 7. Transverse section of lamina ossea (Duckbill) showing a blood-vessel passing up on to the membrane of Reissner. × 150 diameters.

1. Limbus of lamina ossea.
2. Membrane of Reissner.
3. Blood-vessel.
4. Blood-vessel forming a knot.

Fig. 8. Transverse section through middle of the Magpie's cochlea. × 75 diameters.

1. Scala tympani.
2. Ductus cochleæ.
3. Quadrilateral cartilage, (*a*) blood-vessels.

4. Triangular cartilage.
5. Nerve, (*b*) ganglionic cells, (*c*) fibres passing through cartilage.
6. Tegmentum, (*e*) folds.
7. Organ of CORTI, (*f*) membrana basilaris, (*g*) membrana tectoria, (*d*) bone.

POSTSCRIPT.

(Added January 26, 1881.)

To prevent misunderstanding, it may be as well to state that the object of this memoir being the description of the cochlea of the *Ornithorhynchus* and its comparison with that organ in the ordinary Mammal and Bird, I have purposely omitted the description of these last, except where necessary for such comparison.

Very complete descriptions of these will be found in QUAIN'S 'Elements of Anatomy,' 8th edition, vol. ii., for the Mammal; and in PAUL MEYER'S 'Labyrinthe Membraneux du Limaçon chez les Reptiles et les Oiseaux, 1876' (published by J. B. BAILLIÈRE et FILS, Paris), for the Bird and lower vertebrates.

Since presenting this memoir I have discovered a second row of hair cells to the outer side (corresponding to inner side of ordinary Mammals) of the rods of CORTI (see Plate 46, 12, fig. 6, 12).

This is the only animal in which I have seen an additional row of hair cells in this position[*]; nor, indeed, am I aware of such a peculiarity having been noticed by other observers in any form of cochlea.

III. *On the Organization of the Fossil Plants of the Coal-Measures.*—Part XI.

By W. C. WILLIAMSON, *F.R.S., Professor of Botany in the Victoria University.*
(Owens College, Manchester.)

Received June 3,—Read June 17, 1880.

[PLATES 47–54.]

IN his recently-published memoir entitled "Structure Comparée de quelques tiges de la Flore Carbonifère" ('Nouvelles Archives du Muséum, 1879'), M RENAULT says (*loc. cit.*, p. 246) :—"Si les Sigillaires ne sont que l'état plus agé de certains Lépidodendrons, comme le pense M. WILLIAMSON, plusieurs conséquences découleront de cette identité.

"1°. Toute Sigillaire étant représentée dans son très jeune âge par un axe uniquement Lépidodendroide, nous devrons naturellement rencontrer de jeunes Lépidodendrons construits sur autant de types que nous aurons d'axes internes différent de Sigillaires. En effet, en admettant le *Sigillaria vascularis* de M. BINNEY comme type de Sigillaire, nous aurions trois structures différentes pour les axes des jeunes Lépidodendrons correspondants.

"*a.* Cylindre ligneux, représenté par un circle continu de vaisseaux scalariformes, renfermant une moelle centrale parcourue par de nombreux faisceaux vasculaires également scalariformes. (État jeune du *Sigil. vascularis.*)

"*b.* Cylindre ligneux, formé par un cercle continu de vaisseaux scalariformes, entourant une moelle n'offrant aucune trace de vaisseaux. (État jeune des *Diploxylon.*) D'après M. WILLIAMSON, le *L. Harcourtii* serait le représentant de ce type.

"*c.* Cylindre ligneux, formé par une couronne discontinue de faisceaux vasculaires, scalariformes, circonscrivant une moelle dépourvue de faisceaux vasculaires. (État jeune des Sigillaires des genres *Favularia, Leioderma,* et probablement *Rhytidolepis.*) Jusqu'à présent, le second de ces types Lépidendroides seul aurait été recontré."

Having thus defined what he regards as the three distinct types of *Sigillaria* for which corresponding Lepidodendroid branches ought to be found, M. RENAULT proceeds to show what Lepidodendroid types exist according to his views. These are again three in number, all of which he considers to be modifications of the type of *L. Harcourtii* :—

"1°. *Lepidodendron Rhodumnense.*
"2°. „ *Harcourtii.*
"3°. „ *Jutieri.*

"Dans le premier type, les rameaux et la tige sont pourvus d'un axe ligneux, formé de larges vaisseaux rayés sans aucune trace de tissu cellulaire interposé, les plus gros vaisseaux sont au centre, les plus petits à l'extérieur.

"Les cordons foliaires partent toujours de faisceaux vasculaires placés à la periphérie de l'axe ligneux. En dehors on rencontre une écorce formée d'une couche parenchymateuse, puis d'une zone de cellules prosenchyme, qui, dans les vieilles tiges, peut prendre un accroissement considérable, enfin d'une couche cellulaire due en grand partie aux coussinets des cicatrices foliaires.

"Dans le second type, les rameaux et la tige renferment un cylindre vasculaire continu entourant une moelle; du côté de cette dernière, les vaisseaux rayés qui le constituent sont larges, et deviennent plus petits vers le periphérie.

"Les faisceaux vasculaires, qui envoient des cordons aux feuilles sont également placés à la périphérie du cylindre ligneux ; ce cylindre est entouré d'un tissu parenchymateux assez solide, suivi d'un parenchyme corticale un peu plus lâche, enfin d'un zone de cellules étroites plus allongées, disposées en séries régulières."

M. RENAULT's third type, not yet described in detail, will be noticed later. I cannot accept these definitions as in accordance with known facts. I am prepared to demonstrate that we have found the Lepidodendroid representatives of his Sigillarian types *a* and *b*, and that, unless I much mistake, his *Lepidodendron Jutieri* represents *c*. I think that the young growths of the Arran Lepidodendron recorded in my last memoir (Memoir X., Plate 1, figs. 1, 2) correspond with M. RENAULT's *Lepidodendron Rhodumnense* ; in a more advanced stage it represents his *L. Harcourtii* ; and in its matured condition it becomes a Diploxyloid tree.

In addition, M. RENAULT has omitted from his list of Lepidodendroid forms that which BINNEY has named *L. vasculare*, and which, as the following pages will demonstrate, is the Lepidodendroid representative of M. RENAULT's Sigillarian group *a*.

I may further notice another observation, in which M. RENAULT says : "Si certains Lepidodendrons se transforment en Sigillaires par le fait seul de l'âge, c'est à dire par l'apparition plus ou moins tardive d'un bois exogène en dehors de l'axe lépidodendroïde, on doit admettre que l'addition de couches ligneuses ne modifie pas la structure primitive, propre de cet axe, que les faisceaux vasculaires qui ont émis des cordons foliaires ont dû conserver leur disposition particulière et leur structure générale primitive. Les cordons foliaires eux-mêmes déjà formés, et parcourant les feuilles, n'ont pas dû changer de constitution pendant l'accroissement exogène de la tige" (*loc. cit.*, p 247).

I have already shown in my description of the Arran plant, as well as in some measure in that of the Burntisland one (Memoir III., Plate 41, figs. 1, 2, and 3 ; and Plate 42, fig. 9), that such changes do occur ; and I shall now have to advance evidence demonstrating not only that the vasculo-medullary elements, both cellular and vascular, do increase in size and number *after* they have been surrounded by the exogenous zone, but that, in like manner, the foliar vascular bundles are lengthened as the exogenous zone increases in thickness, enabling them still to maintain their connexion with the bases of the leaves that remain attached to the growing stem.

M. RENAULT represents the first of his Sigillarian types by the *Sigillaria vascularis* of Mr. BINNEY, but which is really a state of the common *Lepidodendron selaginoides*. When M. RENAULT denies that a Lepidodendroid state of this supposed Sigillarian plant exists, he overlooks the fact that Mr. BINNEY long ago described and figured a partially decorticated specimen in this condition under the name of *Lepidodendron vasculare*.* In that memoir Mr. BINNEY correctly points out that the plant only differs from his *Sigillaria vascularis* in the absence from the former of an exogenous zone.

Fig. 1 represents a transverse section of a branch of this plant, enlarged about 13 diameters, its actual diameter in the longer direction being nearly ·8. In its centre is the vasculo-medullary axis, *a*, the periphery of which consists of a continuous zone, sharply defined externally, composed of numerous barred vessels. More internally, cellular elements are interposed between these clusters of vessels, until, in the central portion of the axis, the vessels are either isolated or combined in very small groups. The medullary cells are of two kinds, as pointed out in my memoir, Part II., Plate 24, fig. 3. The barred or reticulate character of many of these cells affords an infallible guide to the identification of this plant.

Fig. 2 represents a segment of the only specimen I have seen in which *all* the cortical tissues are preserved; for this specimen I am indebted to Mr. CASH, of Halifax. At *a* we have the periphery of the vascular zone. At *b* is the innermost cortex, separable into three parts—an internal zone, *b*, consisting of very minute cells; a middle area, *b′*, in which the cells are of larger size; and an outer one, *b″*, in which they exhibit a tendency towards a cyclical arrangement. In this zone the foliar vascular bundles, *c*, are invariably intersected, in these transverse sections, at right angles to their *vertical* course. Enclosing this endophloëm we have a middle cortical layer, *d*, which is very rarely preserved, its place being usually vacant, as in fig. 1, *d*. This zone consists of very delicate parenchyma, the cells of which at its inner margin, *d′*, exhibit a disposition towards a radial arrangement. The vascular bundles pass through this zone in an *oblique* direction. The yet more external zone, *e*, is almost invariably preserved in these young branches. It consists of strongly-defined thick-walled cells, which diminish in size from within outwards—the compressed aspect of those shown at fig. 2, *e′*, being obviously accidental. The smaller peripheral cells gradually pass into crescentic clusters of radially disposed prosenchymatous cells, *f*. Fig. 1 demonstrates that these prosenchymatous or bast tissues are first developed in the convex outward extensions of this portion of the bark which underlie the several leaves. The latter organs, composed of coarse parenchyma, are seen at *g*, *g′*. At *g′* is the single foliar bundle; in each of the three leaves of fig. 1 in which this bundle appears there are peculiar condensations, *g''*, of the cellular tissue of the leaf. In fig. 2, *g''*, we find a similar condensation of the peripheral part of the leaf through which the bundle *g'* originally passed.

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* 'Quarterly Journal of the Geological Society of London,' May, 1862, p. 110, plate 6.

Whilst the specimens described are Lepidodendroid branches identical with the type *a* of M. RENAULT's three Lepidodendroid groups, they soon acquire an exogenous vascular zone, *i.e.*, assume a Sigillarian organization. I am indebted to Mr. CASH, Mr. SPENCER, and Mr. BINNS, of Halifax, and to Mr. J. AITKEN, of Urmston, near Manchester, for numerous specimens illustrating this portion of the subject.

Fig. 3 is the vasculo-medullary axis of a specimen like figs. 1 and 2, but with the beginnings of an exogenous zone, *h*. This exogenous growth commenced at a single point at the circumference of the vasculo-medullary axis, from which point it extended laterally in opposite directions, as well as radially, until it enclosed the vasculo-medullary axis in a complete cylinder, composed of vascular wedges separated by numerous medullary rays. In the specimen fig. 3 this growth has begun opposite to the point *x*, and has extended round little more than one-third of the circumference of the medullary axis. Whilst the transverse sections of the vertical portions of the foliar bundles are grouped closely round the axis at *c*, they are separated from it at *c'* by the interposition of the exogenous zone, *h*. Their normal position is in the innermost cortical layer, which is necessarily pushed outwards by the growing exogenous zone.

Fig. 4 is a segment of another section, enlarged 12 diameters, in which a further exogenous development has taken place; and fig. 5 represents the vasculo-medullary axis and inner bark of another section from the same specimen, enlarged 28 diameters. In these specimens the exogenous zone has enclosed fully two-thirds of the circumference of the vasculo-medullary axis. As in fig. 3, the characteristic barred cells of the medulla are conspicuously numerous. The details of the structure of the bark seen in fig. 1 reappear in fig. 4, with the one exception, that the prosenchymatous or bast layer, *f*, instead of filling small detached crescentic areas at the base of each leaf, has now become a thick and continuous zone. The regular continuity of its peripheral border, *f''*, makes it clear that the additions to its thickness have been made to that periphery, and not to its inner margin where it becomes merged in the outer parenchyma, *e*. In more than one of my previous memoirs I have expressed my conviction that this has been the case, and the examples now described appear to confirm those conclusions. It further appears clear that this increase in the thickness of this bast layer is, like the development of the exogenous zone, an indication of a corresponding advance in the age of the stem. The leaf *g*, with its single vascular bundle *g'*, exhibits a similar appearance to what we see in fig. 1, *g'*.

Fig. 6 represents the central axis of another stem, in which the exogenous zone has completely encircled the primary vasculo-medullary axis. A considerable portion of the latter structure has been destroyed, I presume before mineralisation. With these two exceptions, this section is a repetition of fig. 5. The innermost bark, *b*, is now more or less detached from the primary vasculo-medullary axis, and encloses the numerous *transverse* sections of the foliar vascular bundles, *c'*, their more internal portions, *c*, passing almost horizontally through the exogenous zone.[*]

When my memoir, Part II., was written, I was unacquainted with any further advance in the exogenous development of this stem than is seen in the specimens just described; but it is otherwise now. The cabinets of Mr. AITKEN and Mr. CASH have furnished me with several specimens displaying the same stem in various stages of further Diploxyloid growth. Fig. 7 represents one for which I am indebted to Mr. AITKEN, enlarged nearly six diameters. The central vasculo-cellular medullary axis has a mean diameter of about ·25.[†] In another specimen in Mr. AITKEN'S cabinet the diameter of this axis is fully ·4, which is also the diameter of the same structure in a similar section for which I am indebted to Mr. CASH. In a series of specimens in my cabinet, the entire stems of which exhibit a gradual increase in size, I find that in the youngest, in which there is no exogenous growth, the diameter of the vasculo-medullary axis is ·02. In the next, in which the exogenous zone has completed two-thirds of the circle, it is ·05. In another, in which the exogenous zone has completed its circle, it is ·1. In fig. 7 it is ·25, whilst in Mr. AITKEN'S and Mr. CASH'S specimens it has attained to ·4. We thus find that this central axis steadily increases in magnitude with the general growth and age of the branch; and an examination of the specimens shows further that this increase is not due to a mere enlargement of the cells and vessels of which this vasculo-medullary axis is primarily composed, but to an enormous, though gradual, increase in their number. It thus becomes evident that, contrary to the views of M. RENAULT, the cells of the axis retained their genetic activity long after the exogenous zone was developed, and consequently the latter must have been capable of such interstitial changes as allowed it to expand and accommodate itself to the increasing diameter of the structure which it enclosed.

The exogenous zone, *h*, of fig. 7 is composed of very regular, narrow, radiating wedges, undistinguishable, on a casual glance at the transverse section, from the similar structure in a Gymnospermous branch. The medullary rays, *k*, are numerous and large, and nearly all of them contain more or less numerous barred cells that are elongated in the direction of the ray. The comparatively narrow cortical zone *e* is the same as the zone *e* of fig. 1, whilst the prosenchymatous zone *f* is now the dominant one, having more than four times the diameter of *e*, as well as being four times thicker in proportion than the same zone is even in fig. 4. I have frequently had occasion to point out the remarkable tendency displayed, both by the Calamitean and Lepidodendroid plants of the Coal-measures, to the extension of this prosenchymatous zone of the bark, which has here played the part of a periderm.[‡]

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[*] In illustration of these peculiarities of direction, see figs. 33 and 34 of my Memoir IX. The *vertical* direction which they follow, close to the exterior of the vascular medullary axis in the non-exogenous state, is equally transferred to the exterior of the exogenous cylinder when that structure first makes its appearance and when, by age, it has attained to large dimensions.

[†] These measurements are in fractional parts of an inch.

[‡] This portion of the bark in fig. 7 is not devoid of further interest I long ago ascertained that

In my memoir, Part III., Plate 43, fig. 19, I represented a section of a young dichotomising branch of the Burntisland *Lepidodendron,* in its young state, in which the vascular cylinder had divided into two horseshoe-shaped portions, and in fig. 20 of the same plate is shown a similar subdivision of the same plant in the more advanced Diploxyloid stage of growth. Fig. 8 represents a young branch of *Lepidodendron selaginoides* dividing in a similar manner. Each of the two halves of the vascular axis exhibits the usual horseshoe-shaped contour, the innermost cortical cells, *b″*, intervening between the two becoming contiguous to and intermingled with the medullary ones. The whole of the cortical elements and foliar-vascular bundles seen in this specimen are identical with those represented in figs. 5 and 6 ; but whilst at the upper side of the half, A, of the vascular axis we have the crescentic beginnings of an exogenous zone, *h,* no trace of such a zone is seen in the half B. Thus the latter is Lepidodendroid and the former Sigillarian according to the Brongniartian hypothesis. Immediately surrounding the lower half of each axis, especially of B, we find some cells, *b* , arranged in radiating lines, but no trace of vessels. This disposition of cells, occupying the position of a cambium layer, so to arrange themselves is not unimportant.

Having thus traced the history of an example of M. RENAULT'S group *a,* from its Lepidodendroid to its Sigillarian state, I will now deal with his group *b.*

The specimen of *Lepidodendron Harcourtii* described by WITHAM and BRONGNIART was made the foundation of a restored figure by the latter botanist in his 'Végétaux Fossiles.'* The imperfection of the specimen necessarily led to imperfections in the details of the restoration.

M. RENAULT, speaking of *Lepidodendron Harcourtii* as the typical representative of his second group, says,—"On peut s'étonner qu'avec un développement semblable aucune trace de production ligneuse exogène ne se soit pas encore manifestée en dehors du cercle de faisceaux vasculaires d'où partent les cordons foliaires." And again, "On peut donc légitimement conclure que la différence dans le diamètre des rameaux de *Lepidodendron* n'apporte pas de changements dans la disposition générale des tissus que l'on trouve dans ces plantes." On both these points I shall offer evidence that our English specimens exhibit the conditions of which M. RENAULT denies the existence.

Fig. 9 represents a section of a superb example of the young state of *Lepidodendron Harcourtii,* for which I am indebted to Mr. AITKEN. On comparing it with the similar section of *L. selaginoides* it is seen to differ from that plant in two points: first, in the absence of all vessels from the area occupied by the medullary cells, causing the inner boundary of the vascular medullary zone ("étui médullaire" of BRONGNIART) to be sharply defined; and, second, in the much greater uniformity in the size and

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the mineral charcoal, so abundant in almost every British coal of Carboniferous age, is largely composed of small cubical fragments of prosenchymatous bark. Fig. 7, *f,* illustrates the strong tendency of this tissue to be broken up into similar fragments.

* Plate 21, fig. 4.

structure of the smaller cells composing the various layers of the bark. The foliar-vascular bundles also much more frequently retain their barred vessels undestroyed. They appear in the sections as dark, dense bands, instead of being represented by vacant spaces.

No example of *L. Harcourtii* in which all tissues of the stem are preserved has hitherto been described. The mean diameter of the branch figured is about 1½ inches. The cellular medulla, *a*, is invested by the vascular medullary zone *a′*, which, in turn, is enclosed within a dense layer of inner bark *b*, the innermost portion of which is crowded with foliar bundles. Surrounding this is the middle bark *d*, composed of small cells. This layer, which is so rarely preserved in *L. selaginoides*, is as rarely absent from *L. Harcourtii*. The cells of the innermost portion, *e*, of the outer bark are, as is usual in the *Lepidodendra*, somewhat larger, and have thicker walls, than those of the inner bark, whilst the prosenchymatous zone, *f*, appears as a continuous, wavy, but very narrow zone, curving slightly outwards opposite to the base of each leaf, and projected outwards as a sharply defined funnel-shaped prolongation, wherever one of the numerous foliar bundles, *c*, passes through it. The leaves, *g*, differ in no material respect from those of *L. selaginoides*.

Fig. 10 represents part of a transverse section of a branch of rather larger size and somewhat more developed growth.* It exhibits the usual structure of the vascular and cellular portions of the medullary axis as well as of the inner bark. The central portion of the cellular medulla, *a*, has been accidentally destroyed. The vascular medullary cylinder, *a′*, is composed of barred vessels which diminish in size from within outwards. Its peripheral margin presents numerous radiating points, *c′*, which are the bases of as many foliar-vascular bundles. These bend upwards immediately after their orientation, as in *L. selaginoides;* hence they are intersected transversely at *c*, *c*, where they severally occupy the numerous crescentic areas intervening between the projecting points *c′*. The cells of the parenchyma of the innermost portion, *b*, of the inner bark are very minute, usually forming a dense sheath surrounding each foliar bundle *c*. In the middle portion of this inner bark is an irregular layer of cells, *b′*, of somewhat larger size; but its outer zone again consists of smaller cells, *b″*. In this outer zone a narrow band of these small cells appear arranged in radiating lines, *b‴*. This radiating zone bears an important relation to some features to be described. At *d* we have the thick middle bark with its dense foliar bundles, *c″*, *c″*.

Even in this typical *Lepidodendron, when the branches are sufficiently advanced in growth*, there appears a rudimentary zone composed of true radiating, exogenously developed, vascular wedges. Fig. 11 represents a segment of a transverse section of a portion of the zone corresponding to *b″*, *b‴* of fig. 10, enlarged 27 diameters. At *b‴* are the radiating lines of cells already referred to, and at *b″* the outermost part of the zone. At c'', c'' we find two foliar-vascular bundles, belonging to an outer series corresponding to c'' of fig. 10, whilst in the curved line between these two bundles we have a portion of the exogenous zone, composed of numerous radiating wedges, *h*, of barred vessels. At h', h', where these wedges are deflected somewhat from their direct course, their true vascular character is demonstrated. I find this exogenous zone in so many specimens that have attained the right stage of growth as to leave no doubt whatever as to its being a normal feature of the plant. Whether or not we shall hereafter discover it developed into the more perfect Diploxyloid condition seen in other Lepidodendroids remains to be seen. We were long before we found *L. selaginoides* so developed; meanwhile its presence in *L. Harcourtii*, even in this rudimentary form, directly links that plant with the Diploxyloid or Sigillarian group of stems. In many of my specimens some large isolated cells, fig. 11, *b*, appear in the zone b''. In the specimen from which fig. 11 was taken these are larger and more conspicuous than usual.

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* It will be observed that before it begins to develop its exogenous zone this plant attains to much larger dimensions than the *L. selaginoides* does. In this respect it corresponds closely with the Arran plant.

In fig. 9, a'', a considerable segment of the vascular medullary cylinder is detached from the rest, leaving a hiatus in the cylinder, and carrying along with it a semicircular loop of the innermost bark, b', with its associated foliar vascular bundles. In my memoir, Part II., p. 224, I called attention to the fact that the large vascular bundles separated from the main cylinder to supply the tubercles of the *Haloniæ* were given off in the same way, and the bundle a'' has doubtless had some similar destination. Such a division of the primary vascular cylinder differs only in amount from the dichotomous subdivision of a branch illustrated in fig. 9. In another branch of the same stem, contained in the nodule in which the example fig. 9 occurred, I found a similar bundle, of which a representation is given in fig. 12, enlarged 21 diameters. In the branch bundle of fig. 9, the central vasculo-cellular mass, a'', retains the concavo-convex form belonging to it as a small segment of a circle, and the remark applies also to the dark-coloured layer of inner bark, b', which surrounds its convex peripheral border; but in fig. 12, the bundle *a* has assumed a more cylindrical contour; the dark inner bark, b', now completely invests the bundle in the form of a six-sided cylinder, whilst the contiguous cells, *d*, of the middle bark have re-arranged themselves in almost concentric lines around the developing axis. On further examining the central bundle we find that what in fig. 9, a'', was merely a small concavo-convex segment of a large circle, having some cells in the concavity of its inner border, has also changed its form. The extremities of the vascular segment have been bent inwards, converting the segment into a circular cylinder, a', which has enclosed the cells, *a*, that primarily only lay on one side of it.* I am able to demonstrate that this branch, fig. 12, is merely a further development of a structure like fig. 9, a'', b'. Mr. AITKEN kindly allowed me to make a vertical section through the specimen from which the transverse section, fig. 12, was taken. This not only enabled me to trace the bundle downwards into the condition represented in fig. 9, but to a yet lower position where, instead of being detached from the medullary vascular cylinder, the boundary between the two was only indicated in the transverse section by two slight lateral constructions in the otherwise unbroken circle of vessels. Other sections of the same specimen in Mr. Aitken's cabinet exhibit similar conditions. We have here a clear proof that, contrary to the conclusions of M. Renault, age does bring about very important changes in the form and arrangement of the tissues in the branches of these plants. Other examples of stems in a very much younger state, for which I am indebted to Mr. Aitken and Mr. Cash, exhibit corresponding phenomena.

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* The gap left in the vascular cylinder of the parent root stem closes in the same way by the convergence of the separated extremities of the broken circle.

My most recent researches have brought to light another very remarkable series of facts indicating the Lycopodiaceous character of the Lepidodendroid and Sigillarian stems. Whatever else may be doubtful, there is no doubt that *Stigmaria ficoides*, with its peculiar rootlets, is the root alike of *Lepidodendron* and of *Sigillaria*. I had long noticed the circumstance that the vascular bundle in each of these rootlets was rarely in the centre of the small innermost circle of cortical cells by which it is usually surrounded ; but such structures are so often pushed out of their normal position in these fossil plants that I regarded this eccentric position of the rootlet bundle as accidental. The perusal of a very important paper by M. Ph. Van Tieghem,* first called my attention to a peculiarity in the structure of the true roots of the Lycopods, especially of the *Selaginelleæ*, and led me to re-examine the rootlets of *Stigmaria* with the results to be described. M. Van Tieghem has shown that when a rootlet belonging to any class of plants is first formed its primitive vascular elements commence by developing a small vascular bundle at each of two or more points on the periphery of a central cellular circle, from which points each bundle grows by the centripetal additions to the number of the vessels composing it, until the several bundles approach each other centripetally, and thus constitute a symmetrical group of vessels. These primary rootlet bundles are never less than two in number, with the exception of the ultimate rootlets of the *Lycopodiaceæ* and the *Ophioglosseæ*. Each terminal rootlet in these classes of plants contains but one triangular vascular bundle. This bundle first appears as one or two very minute vessels attached to one side of the small cellular cylinder within which it is located. New and larger vessels are subsequently added centripetally to these primary ones, but the bundle always remains eccentric in reference to the cellular sheath within which it is enclosed. The remainder of the cavity within the sheath is occupied by some large thin-walled cells, which M. Van Tieghem regards as the equivalents of the liber of other plants. Fig. 13 represents the central portion of a rootlet of *Selaginella Martensii*, in which *a* indicates the first formed vessels of the vascular bundle, and *a′* the later, centripetally developed and larger ones. The large cells, *b*, are those which M. Van Tieghem identifies with the liber, whilst *c* indicates the next innermost zone of the cortex.

Figs. 14 to 20 exhibit sections of rootlets of *Stigmaria ficoides* in various stages of development. The general relations of the tissues composing these rootlets to their larger exogenous roots were shown in my memoir, Part II., Plate 31, fig. 52. I now only call attention to their relationship to the rootlets of the living *Selaginellæ*. Fig. 14 represents the transverse section of one of these Stigmarian rootlets in the condition in which they most commonly occur. At *a* is a vascular bundle composed of barred vessels enclosed eccentrically within a small cylinder composed of the cells of the innermost cortex *c*. At *d* we have only the outer cortical layer, the intermediate tissues being rarely preserved; but these missing elements have resisted decay in fig. 15, in which example we have the space between the outer cortex *d* and the inner one *c*, occupied by a very delicate cellular tissue *e*, the three layers seen in this specimen apparently corresponding with the three cortical layers *b*, *d*, and *e*, of the branch, fig. 2. The line of demarcation between the outer and middle layers is very sharply defined. Fig. 16 represents a very small rootlet in an early stage of growth; its entire diameter not being more than ·025. Its vascular bundle, *a*, only consists of four very small vessels attached eccentrically to one point of the inner side of the ring of inner cortex, *c*, whilst all that remains of the middle and outer bark is a small band of the former, *e*, and a very thin ring of the latter, *d*. The narrow band of middle bark, *e*, connecting the cylinder of inner-bark cells with the outer cortical zone is of common occurrence in this form. In the specimen figured, some of the cells of the outer zone have become partially detached from the outer ring. It will be observed that the diameter of the entire young rootlet is less than that of the central inner bark cylinder of a matured rootlet represented in fig. 18.

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* "Recherches sur la symmétrie de structure des plantes vasculaires" Par M. Ph. Van Tieghem Annales des Sciences Naturelles.' Cinquième série—Botanique, tome 13, 1870–71.

I fortunately possess proof that these small rootlets belonged to very young roots. The enormous thickness to which the exogenous Stigmarian roots were capable of attaining is well known, and since these thick roots often appear to terminate abruptly like the end of a cucumber, it has been thought by some that they never were of small dimensions. This, however, is an error. The youngest plants can have had no true Stigmarian exogenous root but only rootlets, and the stage of growth at which the exogenous structures first made their appearance has yet to be discovered. It however appears certain that the latter were primarily of very small size, and furnished with proportionally minute rootlets, each of which had but a correspondingly small number of vessels. I have before me a section of the bark of a very small Stigmarian root, from Mr. Aitken's cabinet, the entire diameter of which does not exceed ·2, and which is giving off several rootlets whose average diameter is about ·04. No vascular bundle in the rootlets of this specimen possesses more than from three to five very small vessels. The same cabinet contains a beautiful little section of the exogenous vascular axis of a *Stigmaria*, whose maximum diameter is ·6. This has, of course, been the centre of a larger root than that just described. It gives off numerous *root-bundles* proceeding towards the periphery of the bark, each of which possesses from ten to fourteen vessels. These two specimens are the smallest examples of *Stigmaria* that I have yet met with. On the other hand, in two large Stigmarian exogenous cylinders in my cabinet, each of which has a diameter of fully an inch and a half, I found each of the numerous root-bundles consisting of an average of about forty vessels. These facts show that as the diameter of the exogenously developed root increases with age, this growth is accompanied by a corresponding increase both in the dimensions of each rootlet and in the number and magnitude of the vessels constituting its vascular bundle.

Fig. 17 represents the inner cylinder, *c*, with the vascular bundle, *a*, of a rootlet of the same size as fig. 16, but enlarged 270 diameters. The vascular bundle, *a*, consists of but three small eccentrically placed vessels. The space *b*, occupied in the living *Selaginella* by the large, delicate cells regarded by M. VAN TIEGHEM as liber-cells, (fig. 13, *b*), is here empty. We rarely find a specimen in which this is not the case; even in transverse sections of recent Lycopodiaceous roots these cells frequently become ruptured, and their almost constant absence from the fossil ones is a proof that in the latter these structures were similarly delicate. I find them preserved in a few specimens—as in fig. 15—where they prove to be of much smaller size relatively to the vessels than in the living plants.

Fig. 15 represents a yet larger rootlet, enlarged 37 diameters, in which all the tissues are preserved. The vascular bundle, *a*, has increased to six vessels of larger dimensions, and, as already stated, the ring of inner cortex, *c*, encloses the representatives, *b*, of the "cellules libériennes" of M. VAN TIEGHEM. The outermost cells, *c′*, of the inner cortex exhibit a somewhat defined transition from those of the middle bark, suggesting an homology with the "membrane protectrice" of the French botanist. The cells of the middle bark, *e*, are here well preserved and consist of parenchymatous tissue, very much more delicate and thin-walled than those of either the inner, *c*, or the outer cortex, *d*.

Fig. 14 is a section of a rootlet of the most usual size and aspect. It is enlarged 21 diameters, its actual breadth being about ·2. The vascular bundles, *a*, and their surrounding cylinders of inner bark *c* of two similar rootlets are further enlarged in figs. 18 and 19. We now find that the vessels of fig. 18 have increased to 18, and in fig. 19 to 22 in number. But even this is not the final increase, since I have two rootlets in my cabinet each of which has a diameter of fully ·4. The central portion of one of these is represented in fig. 20, in which the vessels composing the bundle have increased to 38 in number, the last formed ones, *a′*, being much smaller in size than those which immediately preceded them. The first formed ones, *a*, have, in this instance, become accidentally detached from the inner bark cylinder, which is a rather unusual occurrence.

The illustrations I have given will, I think, demonstrate the fact that we have in each of these rootlets a single vascular bundle that commences its development eccentrically in relation to the centre of the rootlet to which it belongs, and that, proceeding from this one pole, it is developed centripetally, the extent of that development, including the number of its constituent vessels, bearing a general relation to the age of the rootlet so far as is indicated by its size. Excepting in the magnitude of its liber-cells, the resemblance to the corresponding organs in the *Selaginelleæ* is complete. Seeing that this peculiar structure only exists in the recent Lycopods and *Ophioglosseæ*, and that no other resemblance exists between the fossil *Lepidodendra* and *Sigillariæ* and the *Ophioglosseæ*, we must fall back upon the Lycopods as the plants with which this form of rootlet indicates true affinities. On forwarding characteristic specimens of these rootlets to M. VAN TIEGHEM he at once replied, expressing his conviction that they belonged to a Lycopodiaceous plant.*

It is now fully established that *Stigmaria ficoides*, to which these rootlets belong, is the root of *Sigillaria*, as well as of *Lepidodendron*. The structure of these rootlets therefore adds an additional link to the long chain of evidence, indicating the close affinity of the *Sigillariæ* with the Lycopods.

But there is yet another feature in these exogenous Lepidodendroid stems which may prove to have some bearing upon the problem of evolution · this is, the presence of numerous barred cells in the medullary rays, to which fact I have already made a slight reference. In my memoir, Part II. (Plate 27, fig. 23, *a*, p. 236), I called attention to the existence of such cells in the triangular spaces that appear to project outwards from the medullary sheath, and are intercalated between the inner extremities of the vascular wedges of the exogenous zone; and in the same memoir (p. 238) I further pointed out that such cells entered into the composition of the medullary rays. This is not only the case with the Arran plant described in my last memoir, but I now discover them to be abundant in the various exogenous states of *Lepidodendron selaginoides*, and to be especially conspicuous in the peripheral layers of the more matured branches. Fig. 21, *k*, represents peripheral portions of two medullary rays of fig. 7, enclosing between them a single wedge of the barred vessels. Fig. 22 represents a portion of another, but similar, section, taken from the point where the exogenous zone, *h*, joins the medullary sheath, *a*. A number of small vessels, *x*, *x*, appear in all sections similar to the latter one, respecting which it is difficult to determine whether they are to be regarded as belonging to the medullary cylinder or to the exogenous zone. Where the medullary rays originate amongst these small vessels, oblong barred cells enter conspicuously into their composition, apparently associated with cells that are not barred. The increase in the number of these barred cells as we follow the course of the ray centrifugally is accompanied by a corresponding increase in its diameter as seen in such transverse sections as fig. 7.

CORDA found similar cells in the medullary rays of his *Diploxylon cycadeoideum*, as pointed out in my memoir, Part II.; but he did not distinguish these rays from foliar bundles, which he believed them all to be. Tangential sections, however, show clearly that this is a mistake, as is obvious from the fact that even in transverse sections these rays are seen to be very much more numerous than the foliar bundles.

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* "Par la structure du cylindre central, et par la division dichotomique qu'on y observe en plusieurs endroits, vos racines appartiennent bien certainement à une Lycopodiacée de la famille des Selaginellées." *In litera*, Oct 30, 1879.

What gives an interest to this frequent occurrence of barred cells in the medullary rays of these exogenous Lycopods is the fact pointed out by DE BARY ('Vergleichende Anatomie,' p. 505), that amongst living plants such cells are chiefly confined to the medullary rays of Gymnosperms and especially to the *Abietinæ*.

GENERAL CONCLUSIONS RESPECTING THE CARBONIFEROUS LYCOPODS.

I think I am justified in expressing my conviction that the present memoir confirms the views I have so long advanced, viz.: that at least many of the Lepidodendroid plants acquire, through advancing age, those characteristics that have hitherto been relied upon to distinguish the Sigillarian from the Lepidodendroid forms. In his latest memoir, already quoted, M. RENAULT combines his fossil Gymnospermous Phanerogams in one group, to which he gives the common name of "Diploxylées," and in a foot-note (*loc. cit.*, p. 260) he says: "Sous ce nom je réunis toutes les plantes qui, dans le faisceau libéro-ligneux de la tige ou des feuilles, présentent un double accroissement, l'un centripète, l'autre centrifuge." My friend here reasserts the old Brongniartian hypothesis in all its definiteness. If the evidence produced in the preceding pages is not sufficient to demonstrate the erroneousness of this hypothesis, I am wholly unable to conceive what kind or number of facts would accomplish that end.

The enormous number of Lepidodendroid and allied plants that have now been described by M. RENAULT and myself seem to me to justify an attempt to ascertain what bearing our discoveries may have upon the problem of Evolution. In making this attempt we must not forget that, whilst our researches have been confined to carboniferous plants, the discoveries of Dr. DAWSON have shown that the differentiation of the Gymnospermous Dadoxylons from the Cryptogamic Lycopods was as definite in the Devonian age as in that which succeeded it. Nevertheless, it seems to have been during the later portion of the Palæozoic epoch that the great changes were effected which caused the arborescent Lycopods to be replaced by the oolitic Gymnosperms; and it fortunately happens that the plants which have furnished so rich a morphological harvest belong to that important period of transition.

In endeavouring to ascertain possible genetic relationships between plants, Geologists have to consider: 1st, external form; 2nd, the successive appearances of different types of elementary tissue; 3rd, the arrangement of those tissues in the several vegetative organs; 4th, the reproductive structures.

The study of external forms has occupied, and still occupies, so many observers that I shall not now dwell upon this subject. Outward resemblances, taken alone, are very unsafe guides, though they are too often the only ones that we can obtain. I would make but two observations in reference to them: So far as external appearances are concerned, I am wholly unable to draw a line between *Lepidodendron* and *Sigillaria*, either as regards bark, leaves, or roots. I would further call attention to the fact that, so far as externals are concerned, the fine *Lycopodium ulicifolium* recently imported from Khasia Hills in northern India in a living state, only requires to be raised upon an arborescent stem to furnish an exact copy of the ancient *Lepidodendra*. Its thick, succulent, dependent branches, devoid of all adventitious roots; its oblong, flat, spirally-arranged leaflets, which leave, when they fall, a distinct Lepidodendroid cicatrix; and its huge, often dichotomous, strobili, usually six inches in length, remind us most vividly of the dependent branches of the Palæozoic Lycopods.

The successive developments of the various vegetable elementary tissues is not unimportant. The modification of spiral tissue known as the "barred" vessel—the "vaisseau rayé" of the French Palæo-botanists, and which approximates closely to the modern "scalariform" tissue—has long been known to be the prevalent vascular element amongst Palæozoic plants. Either in its ordinary shape, or in its "reticulated" modification, it is common to the *Calamites*, the *Sphenophyllums*, the *Asterophyllites*, and the *Lyginodendra*. Amongst our British *Lepidodendra* we have as yet found only the simple barred form; and this statement applies not only to the plants recognised by my opponents as Lycopodiaceous, but also to those Diploxyloid ones for which they claim Gymnospermous rank.

It is a fact of some importance that I have failed to detect in any one of our British fossil Lepidodendroids (including *Sigillariæ*) a trace of those fibres with areolated "margined pits" which exist abundantly in all living Cycads, in which latter, as I showed long ago,[*] modifications of the spiral and the pitted states can be discovered in the same individual vessel.[†]

But M. RENAULT has discovered in France an important group of carboniferous plants which undoubtedly possess the barred and areolated tissues in the same stems, to which group he has given the name of *Poroxylées;* as he points out, these Poroxylons appear to connect the *Sigillariæ* on the one hand with the Gymnospermous *Cordaites* on the other.[‡] This very important discovery, the reality of which there is apparently no reason to doubt, unquestionably links the Palæozoic *Lycopodiaceæ* to the Gymnosperms of the same age, so far as elementary structures are concerned, as completely as any Evolutionist need desire. The arrangement of these elementary tissues into vegetative organs leads us to the same conclusion. The ample development of an exogenous vascular zone in true Cryptogams is demonstrated in the preceding pages as well as in my previous memoirs. In the young *Lepidodendron* the centripetal representative of the vascular medullary sheath, which even BRONGNIART recognized as an "étui médullaire," is not only enormously large, but it is the sole

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[*] "On the Structure and Affinities of some Exogenous Stems from the Coal-measures," 'Monthly Microscopical Journal,' Aug 1, 1869.

[†] It must not be forgotten that Dr DAWSON considers that he has discovered this tissue associated with barred vessels in some of the Canadian *Sigillariæ*.

[‡] *Loc. cit.*, p. 273.

direct link between the roots and the leaves. In a large number of plants the exogenous zone is merely a superadded structure. In the first instance, as in *Lepidodendron Harcourtii,* this latter zone is but feebly represented. Nevertheless this is a first term in a succession which ends in the same zone assuming dimensions that dwarf the relatively narrow, but still cylindrical medullary, sheath of the plants in which the enlarged exogenous growth occurs. In such plants as *Sigillaria spinulosa* and *elegans* the medullary sheath is no longer an unbroken cylinder. It appears as a circle of detached vascular bundles. The exogenous zone now grew more vigorously than the medullary sheath, hence the latter was broken up into separated wedges, leaving intervening spaces through which a direct cellular communication was established between the medulla and the medullary rays of the exogenous zone. Nothing of the kind existed in the lower Lepidodendroid forms. In them the medullary sheath cut off all such direct communication. These changes bring us very near to the narrow medullaiy sheath, composed, as in living Gymnosperms, of spiral vessels, seen in the British *Dadoxylons* and in the St. Étienne *Cordaites;* and, considering how many "missing links" have been discovered during the last five years, I have little doubt but that our continued researches will supply yet further transitional forms, and thus establish the unbroken unity of this chain of vegetable structures.

Hence, though once holding a different view, I am now convinced that so far as we are acquainted with the organisation of the plants of the Coal-measures, such facts as I have briefly referred to are in thorough conformity with the doctrine of Evolution, and that these most characteristic plants can no longer be quoted in opposition to that doctrine.

Our knowledge of the fourth point, viz.: of the organs of reproduction amongst the carboniferous plants, is as yet too imperfect to admit of our employing them as witnesses. They teach most definitely the doctrine of the persistence of types. But the lack of information respecting alike their autogeny and their phylogeny is amply compensated by what we learn from similar organs amongst their living representatives.

I would only add, in conclusion, that whilst I believe I have now demonstrated the transition from the young Lepidodendroid states of M. Renault's two groups *a* and *b* (see page 1) up to their matured Sigillarian conditions, there yet remains his third type of Sigillarian stem *c*, of which no Lepidodendroid state has in his opinion yet been discovered. He says that such a *Sigillaria* would require, according to my views, a Lepidodendroid representative having the following characteristics:—"*c*. Cylindre ligneux formé par une couronne discontinue de faisceaux vasculaires, scalariformes, circonscrivant une moelle dépourvue de faisceaux vasculaires" (*loc. cit*, p. 247); but such a Lepidodendron M. Renault tells us has not yet been found. Yet he announces on page 249 of the same volume that he has found a new Lepidodendron of which he has not yet obtained sufficient knowledge to enable him to say much about it; but he already sees that he can define its broad features as follows:—"Le cylindre ligneux ne serait représenté que par une couronne de faisceaux vasculaires isolés entourant la moelle et d'où partiraient les cordons foliaires. Dans ce type on ne trouverait ni l'axe complètement vasculaire du *L. Rhodumnense* ni le cylindre ligneux continu du *L. Harcourtii*." Surely this plant fulfils the chief requirements of M. RENAULT's third type *c*, his definition of which I have just quoted. But even were it otherwise, the example of the *Lyginodendron Oldhamium* described in my memoir, Part IV., shows that a medullary sheath may be continuous in a young state (see Memoir IV., Plate 22, fig. 2) and yet be broken up into widely separated vascular bundles (*idem*, fig. 3) through advancing age and growth. In these respects the structure of the medullary cylinder in *Sigillaria elegans* and *spinulosa* presents almost a fac-simile of the conditions seen in the larger stems of my *Lyginodendron Oldhamium*.

Calamostachys Binneyana.

Much as has been already accomplished in the investigation of this interesting fruit, we are far from knowing its entire history. I am now able however to fill up two lacunæ in that history. We have hitherto been ignorant of the nature and position of the organic union of the sporangia to the sporangiophores; but I found in the cabinet of Mr. CASH, of Halifax, a transverse section that gives the required information. A portion of this section is shown in fig. 23, *k* being a part of the outer cortex of the strobilus. At *v*, *v*, are two sporangiophores, one of which retains its peripheral, expanded disk at *v'*, and a marginal portion of the other is seen at *v''*. The disk consists of a mass of parenchyma, amongst the cells of which an extension of the bundle of spiral cells that passes along the peduncle of the sporangiophore, is seen prolonged towards the margin of the disk; as the bundle approaches this margin its cells multiply as is the case with the similar structures in the sporangiophores of the recent *Equiseta*, as well as in other very different structures—*e.g.*, the terminations of the hair-like emergences of the *Droseræ*. The peripheral surface of the disk appears to have been composed of a layer of oblong cells, which are planted perpendicularly to it. At *v''* and *v'''* we see that each sporangium is not connected with the peltate end of the sporangiophore by the entire base of the former, as is the case with the living *Equiseta*, but by a very narrow neck of cellular tissue attached to a point a little within the extreme overhanging margin of the sporangiophore; the remainder of the base of the sporangium being entirely free, as is seen at *u'*.

My second discovery is a still more important one. All the examples of this *Calamostachys* hitherto described, possess but one kind of spore. But some years ago I found a fragment, still in my cabinet, in which the sporangia were filled with spores of about three times the diameter of those with which I was familiar. My immediate conviction was that these were macrospores—but the specimen being so fragmentary, and there being no microspores connected with it, I durst not rely upon its apparent indications since it was possible that it might only represent some transitional state of the common form of spore. The correctness of my surmise is however established by the specimen from the Halifax beds represented in fig. 24, and for which I am indebted to Mr. J. AITKEN. It is a slightly oblique and longitudinal section. Hence the sporangia of the upper part have been cut through, almost tangentially, on one side of the vascular axis, whilst the lower ones have been similarly intersected on the other.

Nothing further need be said of the general organization of the plant than that the barren disks with their bracts are seen at *t*, *t′*, whilst the sporangiophores of the fertile verticils are variously intersected at *v*. The importance of the specimen resides in its spores. All the sporangia of the uppermost of the three fertile verticils, as well as those to the right of the middle one, *v*, are filled with the small spores which have now been so often figured and described. The three to the left of the middle verticil, and all the four of the lowermost one, *u′*, contain macrospores. The relative sizes of these two classes of spores are shown in figs. 25 and 26, which are enlarged 220 diameters. The microspores are about ·0031, and macrospores occur as large as ·01, most of them equalling ·0093. The latter exhibit an outer sporangial wall, *a*, as well as an inner one, *b*, whilst a dark-coloured mass, *c*, exists in the centre of most of the examples. The outer wall of fig. 26 has not been intersected on both its surfaces on the plane of its maximum diameter; hence it appears thicker in the figure than it is in reality.

It is scarcely necessary to say that this discovery of macrospores and miscrospores in *Calamostachys Binneyana* supplies another link connecting this strobilus with the *Lycopodiaceæ* in the same measure that it separates the fruit from the *Equisetaceæ*. That no plant belonging to the latter order ever possessed both macrospores and microspores is more than we can venture to affirm; but that no living representative of the group is known to do so is an unquestionable fact—hence to include an heterosporous *Calamostachys* in the Equisetaceous order will involve so large an alteration in the definition of the characteristics of this order as would practically involve the creation of a new one. On the other hand, this discovery strengthens my old conviction that the true affinities of this strobilus are with the *Lycopodiaceæ*. The verticillate arrangements of the fruit, and of what I believe to be the leaves of this plant (*Asterophyllites* or *Sphenophyllum*) constitute no difficulty preventing us from accepting this conclusion. BRONGNIART long ago pointed out how commonly a verticillate foliage occurred amongst living Lycopods.* Fig. 27 represents the best specimen I have seen of a section which has passed tangentially through the margins of three of the horizontal barren disks, *t*, and revealed the form of the bracts, *t′*, which ascend from each of these margins. The specimen is in Mr. AITKEN's cabinet.

Some years ago Mr. CARRUTHERS found a fossil Fungus in a fragment of a *Lepidodendron* from the lower Coal-measures of West Yorkshire, of which he gave a brief account in his annual address to the Geologists' Association for 1876. Mr. BUTTERWORTH, of Oldham, found a second example, which was described and figured by Mr. WORTHINGTON SMITH in the 'Gardeners' Chronicle' for October 20, 1877, under the name of *Peronosporites antiquarius*. Mr. SMITH figures and describes the hyphæ of the Fungus as having septæ, and its supposed oogonia as containing zoospores. The existence of these zoospores was denied by Mr. MURRAY, of the British Museum, in the 'Academy' for November 17, 1877. Still more recently another example of the plant has been met with at Halifax in the cabinet of Mr. SPENCER. Fragments of Lepidodendroid bark, the cells of which are filled with fragmentary hyphæ, but with few traces of oogonia, also from Halifax, are in the cabinet of Mr. CASH. I have had the opportunity of examining all these specimens with the exception of that in Mr. CARRUTHERS' cabinet, which example he informs me is a very imperfect one compared with those more recently discovered.

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* 'Histoire des Végétaux fossiles,' Part 2, pp. 9, 10, plate 7, figs. 1, 7, and 9.

I have failed to find any traces of septa in the hyphæ of this plant, and I quite agree with Mr. MURRAY in his opinion that no zoospores exist in any one of them. Some of the oogonia (fig. 33) contain black coaly matter such as is frequently found in the ordinary cellular tissues of carboniferous plants; but I believe this to be the result of infiltration, since I find it extended into the hollow tubes of some of the hyphæ (as in fig. 33, *a*), and is not confined to the oogonia themselves. Having examined the actual specimen described and figured by Mr. SMITH with the aid of a ZEISS oil-immersion lens, I have had no hesitation in arriving at the same conclusion as Mr. MURRAY has done, viz.: that its oogonia contain no zoospores. The plant has been an unicellular branching mycelium with numerous dilatations on the branching hyphæ, which dilatations seem to have been oogonia. No septal division separates the cavities of these oogonia from the hollow hyphæ prolonged from them.

Fig. 28 is a specimen from Mr. SPENCER'S cabinet. The fungus is here inclosed within the cells, *c*, of a fragment of Lepidodendroid bark. At *a* is a branching mycelium, whilst numerous extremely thin-walled capsules, *b*, are seen either detached or, as at *b'*, connected laterally in a sessile manner with the hypha. Some of the capsules are round, others pyriform and with short hyphoid necks. In this specimen the hyphæ display no traces of septa, neither are there the slightest indications of spores within the capsules.

Fig. 36 represents a portion of the specimen already figured by Mr. SMITH, for the loan of which I am indebted to Mr. YOUNG, F.G.S. Figs. 37 and 37A represent fragments of plants from Mr. SPENCER'S Halifax specimen. Figs. 29 to 33 represent various forms observed in Mr. BUTTERWORTH'S section, and which correspond very closely with the Halifax one. In fig. 29 we have the branching hypha, the branch bearing the base of an oogonium, *a*. In fig. 30 an oogonium, *a*, gives off three hyphoid branches, one of which is not only connected with a second oogonium at *b*, but displays a swelling at *c*, which appears to be the commencement of a third one. In fig. 31, *a*, we have three oogonia connected by two hyphæ; at *c* is the free base of a large oogonium, and at *d* is another, but of smaller size. Fig. 32 again exhibits two oogonia connected by a single hypha, but each oogonium again gives off another hypha from its free extremity. Most of these modifications find their counterparts in the figs. 36, 37, and 37A.

Mr. CASH's cabinet contains several sections of a fragment of cellular bark, most of the cells of which contain broken fungoid hyphæ (figs. 34, 35, and 38), of smaller size than those just described. One of the most perfect of these is represented in fig. 34, *a*. Numerous as are these fragments, indisputable reproductive structures are extremely rare among them; but fig. 34, *b* and *c*, are examples which closely resemble in all but size some of the larger objects already described; and I met with two similar ones, fig. 38, quite equal in their dimensions to those seen in fig. 28. The exceedingly small diameter of the hyphæ in this example, compared with those in figs. 28, 29, and 30, suggest a distinction of species; but we know too little of the effect of changed surroundings upon these simple organisms to make a multiplication of names desirable. This latter form has been described and figured by Mr. CASH in a paper, "On the Fossil Fungi from the lower Coal-measures of Halifax," read before the Yorkshire Geological and Polytechnic in 1879. The specimen was discovered, along with others, by Mr. BINNS, of Halifax.

Mr. SMITH came to the conclusion that since the fossil Fungi which he described possessed, as he believed, septate hyphæ and oogonia containing oospores, they must be ranked with the *Peronosporeæ.* I confess that I cannot confirm the alleged facts nor accept the inference drawn from them. That the plant is a Fungus seems most probable: equally so that its relations are with the *Saprolegnieæ.* The discovery of its reproductive organs in a more perfect state will alone enable us to arrive at a perfectly satisfactory conclusion on this point.

Along with the examples described are numerous single spheres about ·0012 in diameter. It was long doubtful whether or not these belonged to the supposed *Peronosporites*, but I have now no doubt as to their doing so. Their diameter is about the same as that of the oogonia of the branching specimens, and I found one in a small cluster of these objects which had developed into the state corresponding to that represented by fig. 37.

I have occasionally found long branching hyphæ without any dilatations, and which, being associated with the branching ones, I presume belong to the same plant, since they differ in no respect from the shorter hyphæ of the latter. From these facts it would appear that this organism began as a single spore-like cell within the cells and vessels of other plants. This cell threw out from one to three short hyphæ, each of which developed a capsule like an oogonium at its extremity, which, in turn, repeated the same process until a complex branching structure was produced. But under some other circumstances longer branching hyphæ, either devoid of oogonia or only bearing them at much longer intervals, were produced.

In conclusion I have again to thank Mr. CASH and Mr. SPENCER, of Halifax, and Mr. JOHN AITKEN, of Urmston, near Manchester, for the free access they have given me to their cabinets, and to Mr. BUTTERWORTH, Mr. YOUNG, F.G.S. of London, and

Mr. THOMAS BRITTAIN, of Manchester, for sections of the specimen discovered by Mr. BUTTERWORTH.

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PLATE 47.

Lepidodendron selaginoides.

Fig. 1. Transverse section of a young twig of *Lepidodendron selaginoides*, enlarged 13 diameters. *a.* Vasculo-medullary axis. *b.* Innermost cortex. *d.* Zone from which the middle bark has disappeared. *e.* Parenchymatous portion of the outer bark, passing into the bast-layer *f.* *g.* Leaves.

PLATE 51.

Fig. 2. Segment of a similar section to fig. 1. Reference letters as in that figure. Enlarged 27 diameters.

PLATE 48.

Fig. 3. Transverse section of the centre of a twig in which an exogenous vascular zone, *h*, starting from opposite the point *x*, has extended both radially and laterally. *c*, *c′*, foliar vascular bundles in the innermost bark. Enlarged 36 diameters.

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Fig. 6. Central axis of a more advanced twig in which the exogenous growth, *h*, has completely enclosed the vasculo-medullary axis. Enlarged 27 diameters.

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PLATE 49.

Fig. 8. Transverse section of the central portion of a young twig preparing for dichotomous branching. *A*. One portion of the divided vasculo-medullary axis, with an incipient exogenous zone at *h*. *B*. The other portion of the axis, without an exogenous zone, as in figs. 1 and 2. *a*. Medullary cells. *b*. Cells of inner bark. *c*. Foliar vascular bundles. Enlarged 18 diameters.

PLATE 52.

Lepidodendron Harcourtii.

Fig. 9. Transverse section of a young branch 1½ inches in diameter. *a*. Cellular medulla. *a′*. Vasculo-medullary cylinder. *a″*. A segment of *a′* with its surrounding inner cortex, *b′*, becoming separated to supply a lateral branch. *b*. Inner bark. *c*. Foliar vascular bundles. *d*. Middle bark. *e*. Parenchymatous layer of outer bark. *f*. Prosenchymatous, or bast-layer of the same. *g*. Leaves.

PLATE 51.

Fig. 10. Vasculo-medullary axis with the inner and middle bark of a more developed specimen. Enlarged 9 diameters. References as in fig. 9.

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PLATE 51.

Fig. 12. A branch-bundle, similar to fig. 9, *a″*, *b′*, but from nearer to the periphery of a section, and in a more advanced stage of development. *a*, *a′*. Vasculo-medullary axis. *b*. Inner bark. *c*. Foliar vascular bundles. *d*. Middle bark. Enlarged 21 diameters.

PLATE 53.

Stigmarian Rootlets.

Fig. 13. Central portion of a rootlet of the recent *Selaginella Martensii*. *a*, *a′*. Xylem

portion of the root bundle. *b.* Phloëm portion of the same (Liber-cells of VAN TIEGHEM). *c.* Inner cortical cylinder. Enlarged 270 diameters.

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Fig. 18. Vascular bundle and inner bark-cylinder of a rootlet like fig. 14. Enlarged 66 diameters.

Fig. 19. Similar object to fig. 18. Enlarged 140 diameters.

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Medullary rays of Lepidodendron selaginoides.

Fig. 21. Part of fig. 7. Enlarged 200 diameters. *h.* Three vascular wedges. *k, k.* Two medullary rays containing barred cells.

PLATE 54.

Fig. 22. Part of a similar section to fig. 7, but taken from the junction of the exogenous zone, *h, h,* with the vasculo-medullary cylinder *a,* and exhibiting barred cells at the medullary extremities of the medullary rays *k, k.* *x.* Zone of small vessels uniting the vasculo-medullary cylinder with the exogenous wedges *h, h.* Enlarged 200 diameters.

Calamostachys Binneyana.

Fig. 23. Part of a transverse section of a strobilus of *Calamostachys Binneyana* in the plane of a fertile verticil, enlarged 46 diameters. *k.* Outer cortex of the axis of the strobilus. *u.* Sporangia filled with spores. *v.* Sporangiophores. *v″, v‴.* Narrow points near the margins of the peltate peripheral extremities

of the sporangiophores, *v*, at which they are organically united to the sporangia. Enlarged 36 diameters.

Fig. 24. Oblique longitudinal section through a strobilus exhibiting both macrospores and microspores. Enlarged 18 diameters. *k*. Central axis. *t*. Barren bractigerous disks. *t′*. Bracts of *t*. *v*. Sporangiophores of the fertile verticils. *u*. Sporangia with microspores. *u′*. Sporangia with macrospores.

Fig. 25. A microspore of fig. 24, *u*, enlarged 165 diameters.

Fig. 26. A macrospore of fig. 24, *u′*, enlarged 165 diameters. *a*. Exosporium. *b*. Endosporium. *c*. Central mass of carbonised matter.

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Carboniferous Fungi.

Fig. 28. *Peronosporites antiquarius* from Halifax. *a*. Hyphæ. *b*. Oogonia. *c*. Cells of a fragment of bark. Enlarged 375 diameters.

Figs. 29, 30, 31, 32, 33. Various fragments of the same from a specimen from Oldham. Enlarged 500 diameters.

Fig. 34. Fragments of a Fungus with smaller hyphæ from Halifax. Enlarged 675 diameters.

PLATE 48.

Fig. 35. Fragments of a Fungus with smaller hyphæ from Halifax. Enlarged 675 diameters.

Fig. 36. *Peronosporites antiquarius*, Oldham specimen. Enlarged 320 diameters.

Figs. 37, 38. *Peronosporites antiquarius*, Halifax specimens. Enlarged 320 diameters.

During the progress of the investigations recorded in my eleven memoirs I have experienced some difficulties which would not have been felt if I could have examined the originals of the figures and descriptions left by my predecessors in this study. Those who may follow me in these enquiries may be preserved from similar difficulties by knowing that most of the specimens which I have described will find their permanent resting-place in the Geological Museum of the Owens College, Manchester, where they will be accessible to all who may require to consult them.

Manchester, Jan. 23, 1881. W. C. W.

IV. *On the Induction of Electric Currents in Infinite Plates and Spherical Shells.*

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INTRODUCTION.

§ 1. IN Vol. XX. (1872) of the Proceedings of the Royal Society (pp. 160–168) is a beautiful paper by the late Professor CLERK MAXWELL giving an investigation of the induction of currents in an infinite plane sheet of uniform conductivity. For the purposes of the investigation the sheet is supposed infinitely thin; and when it is at rest and influenced by a varying external magnetic system, the effect of the currents induced in it is found to be equivalent to an infinite train of images, at the sheet, of the external system, which, after being formed, move off to infinity with uniform velocity. When the external system revolves uniformly round an axis normal to the sheet, the effect is shown to be the same as if the sheet itself revolved round the axis and the magnetic system remained fixed. The images will then lie in a spiral trail in the form of a helix whose axis is perpendicular to the sheet. This theory was afterwards reproduced in his 'Treatise on Electricity and Magnetism,' and the latter part proved directly from the equations. The analysis there given is somewhat difficult to follow, though it is doubtless possible to present it in a more logically exact form.

The problem of the induction of currents has also been treated by FELICI (TERTOLINI'S 'Annali,' 1853–54) and by JOCHMANN (CRELLE, 1864, and POGG. Ann., 1864). JOCHMANN has solved the case of a sphere which rotates uniformly in a magnetic field symmetrical about the axis of revolution and finds that no currents will be generated in it, but that there will be a certain distribution of free electricity throughout its interior and over its surface. He has also handled the case of an infinite plate of finite thickness, which revolves uniformly round a normal, by neglecting the inductive action of the currents on themselves, and shows that the conditions of the problem may then be satisfied by a system of currents parallel to the faces of the plate; he has also traced the forms of the current and equipotential lines in some simple cases. The solution, however, as MAXWELL has shown in the case of a thin copper disc, can be true only for *very* small values of the angular velocity.

HELMHOLTZ, in an elaborate memoir on the "Equations of Motion of Electricity," in CRELLE'S Journal (vols. 72, 78), has given an exhaustive analysis of the conditions which have to be satisfied in any problem regarding the movement of electricity, and has proved very clearly that the solution of any problem is unique; but he has not dealt with any special case of the problem of induced currents.

MAXWELL'S investigation remains up to the present, so far as I am aware, the only case in which the complete solution of any case of induction has been published.

German writers on current electricity have usually adopted some form of the theory of action at a distance between the elements of different currents, and the free electricity is conceived as a scalar quantity distributed with a certain density throughout the interior and over the surface of conductors. MAXWELL'S theory, which is adopted in the present paper, though it leads generally to similar equations, differs notably from the other in both these respects. The energy is supposed to be seated everywhere in the surrounding medium, and the free electricity is the convergence of a vector quantity termed the electric displacement. The total current, to which electro-magnetic phenomena are due, is compounded of the current of conduction and the time-variations of the electric displacement. Owing to this peculiarity of the theory, the conditions to be satisfied at the surface of separation of two substances will differ from those given by HELMHOLTZ. I have therefore analysed them somewhat fully: taking first, for the sake of generality and the simplicity which it gives, the most general case of two substances in which both the conductivity and specific inductive capacity are to be retained. We can then deduce the conditions at the common surface of two conductors, or of a conductor and a dielectric, which is the case with which we have to do.

One special result of these conditions is that when the vector potential, at the surface, due to all the currents or magnets in the field is at each point perpendicular to the surface of the conductor, the electric potential will vanish everywhere, and there will be no free electricity present either in the conductor or on its surface.

This happens in the case of an infinite plane plate of any thickness. The vector potential (or electro-magnetic momentum) is then everywhere parallel to the surface of the plate, and is derived by vector differentiation from a function P; the current in it is also everywhere parallel to the surface and is derived from a single current function Φ. It also appears that P is the potential of imaginary matter distributed with density Φ; and, during the decay of the currents, P satisfies an equation of the same form as that which regulates the diffusion of heat throughout a solid.

When the plate is infinitely thick there will be no reaction in the inducing system; but when it is very thin, the effect will be that given by MAXWELL as already explained. The general formulæ in this case reproduce his results.

For a solid sphere or shell bounded by two concentric spherical surfaces, the vector potential and current are everywhere at right angles to the radius vector to the common centre, and their values may be derived from two functions, P and Φ, which are related to each other as in a plane plate; during its decay also, P follows the same law as in that case.

When the shell is infinitely thin, the effect, on an external point, of the currents excited in it may be represented by the following system of images, which constitute a generalisation of those of MAXWELL. Divide the time into an infinite number of equal intervals, and at the commencement of any of these let a positive image of the system be formed in the place occupied by its electric image at the surface. Let the parts of this image move towards the centre in straight lines so that the logarithmic decrement of their distances from the centre is constant and equal to $\frac{R}{2\pi a}$ (R being the resistance of the shell and a its radius), and let the intensity of the image increase at each point with a constant logarithmic rate $\frac{R}{4\pi a}$. At the end of the interval let an exactly equal but negative image be formed in the place of the former and move towards the centre in the same manner, and let these operations be repeated at the commencement and end of every interval during which the external system is varying; the action of the sheet on external points will be that due to the above train of images. The action on a point within the sheet may be represented in a somewhat similar manner.

When the shell possesses a finite thickness, or is a solid sphere, it is not possible to express its effect so simply. The variations in the external system produce continually new systems of currents, the law of whose decay may be exhibited by expressing P in a series of terms containing each the product of a tesseral harmonic, a "spherical" function of the radius, and an exponential $e^{-\lambda t}$, the coefficients of which are to be found by known methods.

When the shell degrades into an infinite plate, the "spherical" function becomes an exponential or circular function, and the tesseral harmonic becomes the product of a factor cos $m\phi$ or sin $m\phi$ by a BESSEL'S function $J_m(\kappa\rho)$. The coefficients might then be found by means of NEUMANN'S theorem for expanding $f(x, y)$ in BESSEL'S functions; but their deduction from the corresponding problem of spherical harmonic analysis throws an interesting light upon NEUMANN'S expansion, and especially on the meaning of the symbol ∞ in the limits of integration.

When a symmetrical conductor revolves uniformly about its axis of symmetry for a sufficient length of time, the currents and electric distribution become steady, and the total currents are then identical with the currents of conduction. In the case of a plate or spherical shell, the vector potential and currents are expressible in the same manner as before in terms of two functions P and Φ, which are still related to each other as formerly. The equation which now determines P is $\frac{\sigma}{4\pi}\nabla^2 P = \omega\frac{d(P+P_0)}{d\phi}$.

The general results of calculation verify MAXWELL'S theorem of the spiral trail of images due to an infinitely thin plate. The theorem is also extended to a spherical current sheet; the trail here becomes conical, the locus of points in it, which correspond to a given point of the influencing system, forming a curve which would become portions of an equiangular spiral, if the cone on which it lies were developed on a plane.

The determination of the coefficients in the problems having reference to a sphere or spherical shell depends on the elementary formulæ of reduction of the spherical functions. By adopting a particular mode of constructing the theory of these expressions, it is possible to obtain the necessary properties almost immediately from the definitions: a short sketch of the subject is therefore given, confined, however, to the results which are necessary for the subsequent analysis.

General equations.

§ 2. The general equations of the field in Maxwell's theory are expressed in terms of

The electro-magnetic momentum at a point,	F, G, H
The magnetic induction	a, b, c
The magnetic force	α, β, γ
The total electric current	u, v, w
The current of conduction	p, q, r
The electric displacement	f, g, h
The electromotive force	P, Q, R
The velocity of a point	$\dot{x}, \dot{y}, \dot{z}$

We have also the following scalar quantities,

The electric potential ψ,
The magnetic potential Ω,
The conductivity for electric currents C,
The resistance to conduction, $\sigma=\frac{1}{C}$,
The dielectric inductive capacity K,
The free electricity at any point in the substance of the conductor ϵ per unit of volume, on the surface ϵ' per unit of area.

Since the dielectric surrounding the conductors is air, we may treat the magnetic induction and magnetic force as identical, and we may put

$$a=\frac{dH}{dy}-\frac{dG}{dz}, \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The total electric current u, v, w is connected with the magnetic force by equations of the form,

$$4\pi u=\frac{d\gamma}{dy}-\frac{d\beta}{dz}, \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and satisfies the equation of continuity

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

If we put

$$u=p+\frac{df}{dt}, \ldots, \epsilon=\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz},$$

$$\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}+\frac{d\epsilon}{dt}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

expressing that the loss of electricity by conduction through the faces of an element is equal to the loss of free electricity in the element, a result which may be taken as self evident.

If we put, with MAXWELL, $p=\text{CP}, f=\frac{\text{K}}{4\pi}\cdot\text{P}$, &c., this equation may be written

$$\text{C}\epsilon+\frac{\text{K}}{4\pi}\frac{d\epsilon}{dt}=0.$$

If ϵ_0 be the initial value of ϵ,

$$\epsilon=\epsilon_0 e^{-4\pi\frac{\text{C}}{\text{K}}t}$$

showing that any initial electrification will rapidly disappear in a conductor for which κ is small compared to C. When the substance does not conduct we shall have $\epsilon=\epsilon_0$; so that if we suppose air and other non-conductors initially uncharged, they cannot acquire any charge.

The equations which determine the vector potential or electro-magnetic momentum in terms of the current are

$$\text{F}=\frac{1}{4\pi}\int\frac{u'}{r}dx'dy'dz', \text{ \&c.},$$

where $r=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$, and the integrations are to be extended over all space where there are currents, whether these be currents due to conduction, or time-variations of the electric displacement. F, G, H are thus the potentials of distributions of imaginary matter of finite density; and, therefore, in crossing the surface which divides two substances in the field, we shall have

$$\left.\begin{array}{l} \mathrm{F}=\mathrm{F}', \quad \mathrm{G}=\mathrm{G}', \quad \mathrm{H}=\mathrm{H}' \\ \dfrac{d\mathrm{F}}{d\mathrm{N}}=\dfrac{d\mathrm{F}'}{d\mathrm{N}}, \; \dfrac{d\mathrm{G}}{d\mathrm{N}}=\dfrac{d\mathrm{G}'}{d\mathrm{N}}, \; \dfrac{d\mathrm{H}}{d\mathrm{N}}=\dfrac{d\mathrm{H}'}{d\mathrm{N}}, \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

F′, G′, H′ being the components of the electro-magnetic momentum in the second substance, and $d\mathrm{N}$ an element of the normal drawn from the first into the second.

The other boundary conditions are more difficult to recognise. Let us first confine our attention to the common surface of separation of two substances at rest. In the first the equations of electromotive force are

$$\left.\begin{array}{l} \mathrm{P}=-\dfrac{d\mathrm{F}}{dt}-\dfrac{d\psi}{dx} \\ \mathrm{Q}=-\dfrac{d\mathrm{G}}{dt}-\dfrac{d\psi}{dy} \\ \mathrm{R}=-\dfrac{d\mathrm{H}}{dt}-\dfrac{d\psi}{dz} \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

with similar equations for the second. Let l, m, n be the direction cosines of the normal (N) at any point, and let $\mathfrak{F}=l\mathrm{F}+m\mathrm{G}+n\mathrm{H}$; then, since

$$u=\left(\mathrm{C}+\frac{\mathrm{K}}{4\pi}\frac{d}{dt}\right)\mathrm{P}, \text{ \&c.},$$

$$lu+mv+nw=-\left(\mathrm{C}+\frac{\mathrm{K}}{4\pi}\frac{d}{dt}\right)\frac{d\mathfrak{F}}{dt}-\left(\mathrm{C}+\frac{\mathrm{K}}{4\pi}\frac{d}{dt}\right)\frac{d\psi}{d\mathrm{N}};$$

and, for the second substance,

$$lu'+mv'+nw'=-\left(\mathrm{C}'+\frac{\mathrm{K}'}{4\pi}\frac{d}{dt}\right)\frac{d\mathfrak{F}}{dt}-\left(\mathrm{C}'+\frac{\mathrm{K}'}{4\pi}\frac{d}{dt}\right)\frac{d\psi'}{d\mathrm{N}}.$$

The condition of "continuity" at the surface requires that

$$lu+mv+nw=lu'+mv'+nw',$$

and therefore

$$\left\{\mathrm{C}-\mathrm{C}'+\left(\frac{\mathrm{K}-\mathrm{K}'}{4\pi}\right)\frac{d}{dt}\right\}\frac{d\mathfrak{F}}{dt}+\left(\mathrm{C}+\frac{\mathrm{K}}{4\pi}\frac{d}{dt}\right)\frac{d\psi}{d\mathrm{N}}-\left(\mathrm{C}'+\frac{\mathrm{K}'}{4\pi}\frac{d}{dt}\right)\frac{d\psi'}{d\mathrm{N}}=0 \quad . \; . \; . \quad (6)$$

The free electricity ϵ' at the surface is given by

$$\begin{aligned} 4\pi\epsilon' &= \mathrm{K}\left(-\frac{d\mathfrak{F}}{dt}-\frac{d\psi}{d\mathrm{N}}\right)-\mathrm{K}'\left(-\frac{d\mathfrak{F}}{dt}-\frac{d\psi'}{d\mathrm{N}}\right) \\ &= (\mathrm{KC}'-\mathrm{K}'\mathrm{C})\left(\mathrm{C}-\mathrm{C}'+\left(\frac{\mathrm{K}-\mathrm{K}'}{4\pi}\right)\frac{d}{dt}\right)^{-1}\left(\frac{d\psi}{d\mathrm{N}}-\frac{d\psi'}{d\mathrm{N}}\right) \quad . \; . \; . \; . \quad (6,) \end{aligned}$$

(*a.*) When both the substances are good conductors, we may take $K=K'=0$; the equations then become

$$\left.\begin{aligned}(C-C')\frac{d\mathfrak{F}}{dt}+C\frac{d\psi}{dN}-C'\frac{d\psi'}{dN}&=0\\ \epsilon'&=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(*b.*) When the first substance is a conductor and the second a non-conductor (air), we shall have $K=0$, $C'=0$; the equations are then, electro-magnetic measurements being still employed,

$$\left.\begin{aligned}\left(C-\frac{K'}{4\pi}\frac{d}{dt}\right)\frac{d\mathfrak{F}}{dt}+C\frac{d\psi}{dN}-\frac{K'}{4\pi}\frac{d}{dt}\frac{d\psi'}{dN}&=0\\ 4\pi\epsilon'=-K'C\left(C-\frac{K'}{4\pi}\frac{d}{dt}\right)^{-1}\left(\frac{d\psi}{dN}-\frac{d\psi'}{dN}\right)&\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

But K' is infinitely small compared to C; and therefore, if we write (ϵ') for the electrostatic measure of ϵ', we shall have

$$\left.\begin{aligned}\frac{d\mathfrak{F}}{dt}+\frac{d\psi}{dN}&=0\\ 4\pi(\epsilon')&=\frac{d\psi'}{dN}-\frac{d\psi}{dN}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We may derive an important corollary from these results.

If $\mathfrak{F}$ is always $=0$ at the surface of the conductors (as will be the case in the following problems), we shall have

$$\frac{d\psi}{dN}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

But from equations (5), remembering that $\epsilon=0$, we derive

$$\nabla^2\psi=0, \text{ where } \nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

within the conductor, therefore, ψ is everywhere zero; and since ψ' always satisfies the equation $\nabla^2\psi'=0$ outside the surface and is zero at every point of it, it follows that ψ' is also everywhere zero, and there is no free electricity either within or upon the conductor.

Let us now collect the results of the above discussion, so far as it relates to a conductor disturbed by the introduction or variation of a magnetic system in a space surrounded by air; putting $K=0$, and writing σ for $\frac{1}{C}$, (1.) Within the conductor

$$\frac{\sigma}{4\pi}\nabla^2 F=-\frac{dF}{dt}-\frac{d\psi}{dx}$$

$$\frac{\sigma}{4\pi}\nabla^2 G=-\frac{dG}{dt}-\frac{d\psi}{dy}$$

$$\frac{\sigma}{4\pi}\nabla^2 H=-\frac{dH}{dt}-\frac{d\psi}{dz}$$

$$\epsilon=0,\quad \nabla^2\psi=0.$$

(2.) Outside the conductor $C'=0$, and K' may be neglected; therefore

$$\nabla^2 F'=0,\quad \nabla^2 G'=0,\quad \nabla^2 H'=0,$$

$$\epsilon''=0,\quad \nabla^2\psi'=0,$$

(ϵ''=electric density outside).

(3.) At the surface of the conductor

$$F=F',\ \frac{dF}{dN}=\frac{dF'}{dN},\ \&c.$$

$$\frac{d\mathfrak{F}}{dt}+\frac{d\psi}{dN}=0,\ \mathfrak{F}=lF+mG+nH,\ \psi=\psi'$$

$$4\pi(\epsilon')=\frac{d\psi'}{dN}-\frac{d\psi}{dN}.$$

(4.) When $\mathfrak{F}=0$, $\psi=0$, $\psi'=0$, $\epsilon'=0$, and the currents are confined to the conductor.

When the conducting substance is moving in any manner, the equations of electromotive force are (MAXWELL, Vol. II., Art. 598)

$$P=\gamma\dot{y}-\beta\dot{z}-\frac{dF}{dt}-\frac{d\psi}{dx}$$

$$Q=\alpha\dot{z}-\gamma\dot{x}-\frac{dG}{dt}-\frac{d\psi}{dy}$$

$$R=\beta\dot{x}-\alpha\dot{y}-\frac{dH}{dt}-\frac{d\psi}{dz}$$

If we differentiate these with respect to x, y, z and add, putting for $\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}$ its electrostatic value $4\pi(\epsilon)$, we find

$$4\pi(\epsilon)=\dot{x}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\ldots+\alpha\left(\frac{d\dot{z}}{dy}-\frac{d\dot{y}}{dz}\right)+\ldots-\nabla^2\psi$$

$$=-\frac{1}{4\pi}(u\dot{x}+v\dot{y}+w\dot{z})+2(\alpha\dot{\omega}_1+\beta\omega_2+\gamma\dot{\omega}_3)-\nabla^2\psi\ .\quad .\quad .\quad .\quad (12)$$

where $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$ are the angular velocities of the element at x, y, z. We cannot, therefore, look upon ψ as the potential of free electricity; in fact it is easy to see that there will be no free electricity inside it, just as when it was at rest. If the conductor be symmetrical about an axis, and revolve about it sufficiently long for the currents to become steady, the total current will become identical with the conduction current, and there will be no flow across the surface. We may then take at every point of the surface

$$lu+mv+nw=0.$$

The triple integrals in the expressions for F, G, H are then to be taken only throughout the conductor, and we shall have as before F and $\frac{dF}{dN}$, &c., continuous in passing across the surface; and outside, F, G, H satisfy the equations

$$\nabla^2 F=0,\ \nabla^2 G=0,\ \nabla^2 H=0.$$

The special problems which are solved in the present paper depend for their solution on certain properties of the vector potential; and it will therefore not be out of place to devote a little space to their preliminary discussion. We shall thereby gain a clearer insight into the subsequent analysis.

The vector potential.

§ 3. (A.) We shall first examine the nature of the vector potential inside a space due to magnets or currents outside that space. It is connected with the magnetic potential Ω by the equations.

(*a.*)

$$\left.\begin{aligned}\frac{dH}{dy}-\frac{dG}{dz}&=-\frac{d\Omega}{dx}\\ \frac{dF}{dz}-\frac{dH}{dx}&=-\frac{d\Omega}{dy}\\ \frac{dG}{dx}-\frac{dF}{dy}&=-\frac{d\Omega}{dz}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and it is clear that, if we can find one set of values F, G, H satisfying the equations when Ω is supposed given, the complete values will be

$$F+\frac{d\chi}{dx},\ G+\frac{d\chi}{dy},\ H+\frac{d\chi}{dz},$$

where χ is an arbitrary function of x, y, z. It is therefore necessary for our purpose to find one solution only.

If we can express Ω in the form

$$\Omega=\left(A\frac{d}{dx}+B\frac{d}{dy}+C\frac{d}{dz}\right)P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where P satisfies the equation $\nabla^2P=0$, the equations are satisfied by

$$\left.\begin{aligned}F&=\left(B\frac{d}{dz}-C\frac{d}{dy}\right)P\\ G&=\left(C\frac{d}{dx}-A\frac{d}{dz}\right)P\\ H&=\left(A\frac{d}{dy}-B\frac{d}{dx}\right)P\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and, in particular, if

$$\left.\begin{aligned}&\Omega=\frac{dP}{dz}\\ &F=-\frac{dP}{dy},\ G=\frac{dP}{dx},\ H=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

These expressions (15) may be easily verified by actual differentiation, taking account of $\nabla^2P=0$, and may be looked upon as a generalisation of the equations given by MAXWELL ('Electricity,' Art. 405). They give additional interest to his expression for a solid negative harmonic

$$\left(A_1\frac{d}{dx}+B_1\frac{d}{dy}+C_1\frac{d}{dz}\right)\ldots\left(A_n\frac{d}{dx}+B_n\frac{d}{dy}+C_n\frac{d}{dz}\right).\frac{1}{r};$$

and in forming F, G, H any one of the n factors may be chosen to furnish A, B, C; the results obtained by taking two different factors differing by quantities of the form $\frac{d\chi}{dx}, \frac{d\chi}{dy}, \frac{d\chi}{dz}$, as I have verified. When Ω is a solid harmonic of positive degree, it is more convenient to use the simplified form (16), and in the case of a tesseral solid harmonic, the results of differentiation give rise to a series of very interesting theorems, which, however, do not interest us at present.

It will be observed that the expressions given for F, G, H satisfy the equation of no convergence

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0.$$

(*b.*) When we use semipolar coordinates ρ, ϕ, z, given by

$$x=\rho\cos\phi,\ y=\rho\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

the equations (13) transform into

$$\left.\begin{aligned}\frac{d\mathrm{H}}{\rho d\phi}-\frac{d.\rho\mathrm{G}}{\rho dz}&=-\frac{d\Omega}{d\rho}\\ \frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{d\rho}&=-\frac{d\Omega}{\rho d\phi}\\ \frac{d.\rho\mathrm{G}}{\rho d\rho}-\frac{d\mathrm{F}}{\rho d\phi}&=-\frac{d\Omega}{dz}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(18)$$

Here F, G, H are now the resolved components of F, G, H in the ρ—, ϕ—, z— directions, and the equations are obtained by resolving the two sides of equations (13) in these directions. The expressions on the left hand sides of the above equations are most readily recognised by observing that, when they vanish, we must have

$$[\mathrm{F}dx+\mathrm{G}dy+\mathrm{H}dz=]\,\mathrm{F}d\rho+\mathrm{G}\rho d\phi+\mathrm{H}dz$$

an exact differential. The equations themselves are satisfied by

$$\left.\begin{aligned}&\Omega=\frac{d\mathrm{P}}{dz},\quad \nabla^2\mathrm{P}=0,\\ &\mathrm{F}=-\frac{1}{\rho}\frac{d\mathrm{P}}{d\phi},\ \mathrm{G}=+\frac{d\mathrm{P}}{d\rho},\ \mathrm{H}=0\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(19)$$

and the condition of no convergence

$$\frac{1}{\rho}.\frac{d.\rho\mathrm{F}}{d\rho}+\frac{d\mathrm{G}}{\rho d\phi}+\frac{d\mathrm{H}}{dz}=0,\text{ is satisfied.}$$

(*c.*) If we employ polar coordinates r, θ, ϕ

$$[\mathrm{F}dx+\mathrm{G}dy+\mathrm{H}dz]=\mathrm{F}dr+\mathrm{G}rd\theta+\mathrm{H}.r\sin\theta d\phi$$

and the results of transforming the coordinates are obviously

$$\left.\begin{aligned}\frac{1}{r^2\sin\theta}\left(\frac{d.r\sin\theta\mathrm{H}}{d\theta}-\frac{d.r\mathrm{G}}{d\phi}\right)&=-\frac{d\Omega}{dr}\\ \frac{1}{r\sin\theta}\left(\frac{d\mathrm{F}}{d\phi}-\frac{d.r\sin\theta\mathrm{H}}{dr}\right)&=-\frac{d\Omega}{rd\theta}\\ \frac{1}{r}\left(\frac{d.r\mathrm{G}}{dr}-\frac{d\mathrm{F}}{d\theta}\right)&=-\frac{d\Omega}{r\sin\theta d\phi}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(20)$$

These equations are satisfied by

$$\left.\begin{aligned}\Omega&=\frac{d}{dr}(r\mathrm{P}),\ \nabla^2\mathrm{P}=0\\ \mathrm{F}&=0,\ \mathrm{G}=-\frac{d\mathrm{P}}{\sin\theta d\phi},\ \mathrm{H}=+\frac{d\mathrm{P}}{d\theta}\end{aligned}\right\}\quad\ldots\ldots\quad(21)$$

and may be readily verified by substitution and actual differentiation, as in the former cases, observing that $\nabla^2\mathrm{P}=0$. They coincide with the expressions given by **Maxwell** (Vol. II., Art. 671), and before him by **Jochmann**; they likewise satisfy the condition of no convergence, which in this case is

$$\frac{1}{r^2}\frac{d.\mathrm{F}r^2}{dr}+\frac{1}{r\sin\theta}\frac{d.\sin\theta\mathrm{G}}{d\theta}+\frac{d\mathrm{H}}{r\sin\theta d\phi}=0.$$

(B.) The vector components of the electro-magnetic momentum give rise to the vector potential of the magnetic force in the space in which the currents themselves exist: we shall therefore enquire what distribution of currents must be assigned that the vector potentials due to them may be respectively of the foregoing types.

(*a.*) *Rectangular coordinates.*

The equations to be satisfied are

$$4\pi u=\frac{d\gamma}{dy}-\frac{d\beta}{dz},\ \ldots$$

$$\alpha=\frac{d\mathrm{H}}{dy}-\frac{d\mathrm{G}}{dz},\ \ldots$$

If we take

$$\mathrm{F}=-\frac{d\mathrm{P}}{dy},\ \mathrm{G}=\frac{d\mathrm{P}}{dx},\ \mathrm{H}=0\quad\ldots\ldots\quad(22_1)$$

we find that the above equations are satisfied by

$$\left.\begin{aligned}&\alpha=-\frac{d^2\mathrm{P}}{dxdz},\ \beta=-\frac{d^2\mathrm{P}}{dydz},\ \gamma=\frac{d^2\mathrm{P}}{dx^2}+\frac{d^2\mathrm{P}}{dy^2}\\ &u=-\frac{d\Phi}{dy},\ v=\frac{d\Phi}{dx},\ w=0\end{aligned}\right\}\quad\ldots\ldots\quad(22_2)$$

where

$$4\pi\Phi=-\nabla^2\mathrm{P}.$$

Φ is here the current function, and $\nabla^2\mathrm{P}$ is not now supposed to be equal to zero; in fact, nothing is at present supposed to be known about it.

(*b.*) *Semipolar coordinates.*

The $x-$ and $y-$ directions are variable and are supposed to be in the directions of

$d\rho$ and $d\phi$ respectively, the directed quantities, as above, preserving their former designations.

Our equations are

$$4\pi u=\frac{1}{\rho}\left(\frac{d\gamma}{d\phi}-\frac{d\rho\beta}{dz}\right),\quad 4\pi v=\frac{d\alpha}{dz}-\frac{d\gamma}{d\rho},\quad 4\pi w=\frac{1}{\rho}\left(\frac{d\rho\beta}{d\rho}-\frac{d\alpha}{d\phi}\right)$$

$$\alpha=\frac{1}{\rho}\left(\frac{d\mathrm{H}}{d\phi}-\frac{d\rho\mathrm{G}}{dz}\right),\quad \beta=\frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{d\rho},\quad \gamma=\frac{1}{\rho}\left(\frac{d\rho\mathrm{G}}{d\rho}-\frac{d\mathrm{F}}{d\phi}\right),$$

and we may satisfy these by

$$\mathrm{F}=-\frac{1}{\rho}\frac{d\mathrm{P}}{d\phi},\ \mathrm{G}=\frac{d\mathrm{P}}{d\rho},\ \mathrm{H}=0 \quad \ldots\ldots\ldots \quad (23_1)$$

and

$$\left.\begin{array}{l}\alpha=-\dfrac{d^2\mathrm{P}}{d\rho dz},\ \beta=-\dfrac{1}{\rho}\dfrac{d^2\mathrm{P}}{d\phi dz},\ \gamma=\dfrac{1}{\rho}\dfrac{d}{d\rho}\left(\rho\dfrac{d\mathrm{P}}{d\rho}\right)+\dfrac{1}{\rho^2}\dfrac{d^2\mathrm{P}}{d\phi^2}\\ u=-\dfrac{1}{\rho}\dfrac{d\Phi}{d\phi},\ v=\dfrac{d\Phi}{d\rho},\ w=0\end{array}\right\} \quad \ldots\ldots \quad (23_2)$$

$$4\pi\Phi=-\nabla^2\mathrm{P};$$

here

$$\nabla^2=\frac{1}{\rho}\frac{d}{d\rho}\cdot\rho\frac{d}{d\rho}\cdot+\frac{1}{\rho^2}\frac{d^2}{d\phi^2}\cdot+\frac{d^2}{dz^2}\cdot$$

(*c.*) *Polar coordinates.*

The equations in this case are, resolving along dr, $rd\theta$, $r\sin\theta d\phi$,

$$4\pi u=\frac{1}{r\sin\theta}\left(\frac{d\gamma\sin\theta}{d\theta}-\frac{d\beta}{r\phi}\right),\quad 4\pi v=\frac{1}{r\sin\theta}\left(\frac{d\alpha}{d\phi}-\frac{dr\gamma\sin\theta}{dr}\right),\quad 4\pi w=\frac{1}{r}\left(\frac{d.r\beta}{dr}-\frac{d\alpha}{d\theta}\right)$$

$$\alpha=\frac{1}{r\sin\theta}\left(\frac{d\mathrm{H}\sin\theta}{d\theta}-\frac{d\mathrm{G}}{d\phi}\right),\ \ldots=\ \ldots \qquad \ldots=\ \ldots \quad ;$$

and we may put in these

$$\mathrm{F}=0,\ \mathrm{G}=-\frac{1}{\sin\theta}\frac{d\mathrm{P}}{d\phi},\ \mathrm{H}=\frac{d\mathrm{P}}{d\theta} \quad \ldots\ldots\ldots \quad (24_1)$$

$$\left.\begin{array}{c}\alpha=+\dfrac{1}{r}\left[\dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\cdot\sin\theta\dfrac{d\mathrm{P}}{d\theta}+\dfrac{1}{\sin^2\theta}\dfrac{d^2\mathrm{P}}{d\phi^2}\right],\ \beta=-\dfrac{1}{r}\dfrac{d^2(\mathrm{P}r)}{drd\theta},\ \gamma=-\dfrac{1}{r\sin\theta}\dfrac{d^2(\mathrm{P}r)}{drd\phi}\cdot\\ u=0,\ v=-\dfrac{d\Phi}{\sin\theta d\phi},\ w=+\dfrac{d\Phi}{d\theta}\end{array}\right\} \ . \quad (24_2)$$

where

$$4\pi\Phi=-\nabla^2\mathrm{P}.$$

General problem of induction.

§ 4. The problems to which we shall chiefly devote our attention in what follows are the induction of currents in a plate (i.) of finite and (ii.) of infinitesimal thickness, either at rest or revolving uniformly about an axis normal to its faces, and in a sphere or spherical shell either of finite or infinitesimal thickness under the same conditions.

We may observe however, preliminarily, that *any* solution of any problem of induction which satisfies all the conditions of the question is the only solution that can exist. For all the equations being linear the difference between the functions which express two solutions would also satisfy all the differential equations, and would correspond to the case in which the conductor is under no external inducing forces, and therefore no currents can be set up in it.

This observation has been made by MAXWELL and has been established more in detail by HELMHOLTZ in his memoir. From the linearity of the equations we may also conclude that the effect of different systems acting either simultaneously or in succession may be found by adding the effects due to each separately.

In particular, when the conductor is at rest, its state at any time t may be considered as the aggregate effect produced by a continuous series of impulses during its previous history. The effect of each of these impulses, after being received, decays according to a certain law, and each contributes a certain amount to the total result at any time. The characteristic equation in these investigations is

$$\frac{\sigma}{4\pi}\nabla^2 P=\frac{d}{dt}(P+P_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where P_0 is due to the direct action of the external electro-magnets.

Let an impulsive change take place in this external system, by which P_0 suddenly rises from 0 to P'_0, and let P increase at the same time from 0 to P′; then, integrating the above equation during the impulse, we obtain

$$P'+P'_0=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The impulse having been administered, P decays according to a law peculiar to each system, and satisfies during its decay the equation

$$\frac{\sigma}{4\pi}\nabla^2 P=\frac{dP}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Let $P_0(t)$ be the value of P_0 at any time t: this value may be supposed to have been accumulated by impulses during the previous history of the system, and, if τ be the time measured backwards from t, we may express this fact by the equation

$$P_0=-\int_0^\infty \frac{dP_0(t-\tau)}{d\tau}d\tau.$$

Let $\chi_t(P_0)$ represent the law of decay of a system which was initially represented by P_0; then, combining equations 26 and 27, we see that the complete value of P at any time for the system is given by

$$P=+\int_0^\infty \chi_\tau\left(\frac{dP_0(t-\tau)}{d\tau}\right).\,d\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

To complete the solution, therefore, we have only to determine the nature of χ_τ.

Case of an infinite plate of finite thickness.

§ 5. Let us now take up the case of an infinitely extended plane plate of thickness $2b$, and suppose the origin somewhere midway between the two faces of the plate so that its faces are determined by $z=\pm b$.

The scalar and vector potentials of external magnets or currents may be denoted by

$$\Omega_0=\frac{dP_0}{dz},\ F_0=-\frac{dP_0}{dy},\ G_0=\frac{dP_0}{dx},\ H_0=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The equations of the currents in the plate will have for their type

$$\sigma u=-\frac{d(F+F_0)}{dt}-\frac{d\Psi}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and we shall prove that all the conditions of the problem may be satisfied by taking

$$\left.\begin{array}{l}\Psi=0,\ F=-\dfrac{dP}{dy},\ G=\dfrac{dP}{dx},\ H=0\\ u=-\dfrac{d\Phi}{dy},\ v=\dfrac{d\Phi}{dx},\ w=0\\ \dfrac{\sigma}{4\pi}\Phi=-\nabla^2P\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The equations of electromotive force are now reduced to the single characteristic equation

$$\frac{\sigma}{4\pi}\nabla^2P=\frac{d}{dt}(P+P_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Since there is no free electricity anywhere present, the currents in the plate are closed currents, and therefore F, G, H and their differential coefficients are all continuous in passing across the boundary of the conductor. All these conditions will be satisfied by having

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$$\left.\begin{aligned} \mathrm{P}_{\text{outside}} &= \mathrm{P}_{\text{inside}} \\ \left(\frac{d\mathrm{P}}{dz}\right)_{\text{outside}} &= \left(\frac{d\mathrm{P}}{dz}\right)_{\text{inside}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

when $z=\pm b$; and it must be remembered that, outside the plate, the vector potential is determined by

$$\mathrm{F}=-\frac{d\mathrm{P}}{dy},\ \mathrm{G}=\frac{d\mathrm{P}}{dx},\ \mathrm{H}=0,\ \nabla^2\mathrm{P}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

If we employ semipolar coordinates the equation in P in the substance of the plate may be written

$$\frac{\sigma}{4\pi}\left(\frac{d^2\mathrm{P}}{d\rho^2}+\frac{1}{\rho}\frac{d\mathrm{P}}{d\rho}+\frac{1}{\rho^2}\frac{d^2\mathrm{P}}{d\phi^2}+\frac{d^2\mathrm{P}}{dz^2}\right)=\frac{d\mathrm{P}}{dt}.$$

To satisfy it, put

$$\mathrm{P}=\cos m\phi \mathrm{J}_m(\kappa\rho)(\mathrm{A}\cos nz+\mathrm{B}\sin nz)e^{-\lambda t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where $\mathrm{J}_m(\kappa\rho)$ is BESSEL's function of the m^{th} degree satisfying the equation

$$\frac{d^2\mathrm{J}}{d\rho^2}+\frac{1}{\rho}\frac{d\mathrm{J}}{d\rho}+\left(\kappa^2-\frac{m^2}{\rho^2}\right)\mathrm{J}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and

$$\lambda=\frac{\sigma}{4\pi}(\kappa^2+n^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

We observe also that κ is a constant, which the problem does not enable us to determine: it must therefore be supposed to have all values from 0 to ∞; m is necessarily a positive integer.

Outside the plate P is given by $\nabla^2\mathrm{P}=0$, and is satisfied by

$$\left.\begin{aligned} \mathrm{P}&=e^{-\lambda t}\cos m\phi \mathrm{J}_m(\kappa\rho).\mathrm{C}e^{-\kappa z},\ z \text{ positive} \\ \mathrm{P}&=e^{-\lambda t}\cos m\phi \mathrm{J}_m(\kappa\rho).\mathrm{D}e^{+\kappa z},\ z \text{ negative} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

it being observed that P must vanish when $z=\infty$.

To determine n, we have, by equations (33),

(1) when $z=+b$.

$$\begin{aligned} \mathrm{A}\cos nb+\mathrm{B}\sin nb &= \mathrm{C}e^{-\kappa b} \\ -n(\mathrm{A}\sin nb-\mathrm{B}\cos nb) &= -\kappa\mathrm{C}e^{-\kappa b}, \end{aligned}$$

(2) when $z=-b$

$$\begin{aligned} \mathrm{A}\cos nb-\mathrm{B}\sin nb &= \mathrm{D}e^{-\kappa b} \\ +n(\mathrm{A}\sin nb+\mathrm{B}\cos nb) &= \kappa\mathrm{D}e^{-\kappa b}. \end{aligned}$$

Eliminating C and D,

$$A(\kappa\cos nb - n\sin nb) + B(\kappa\sin nb + n\cos nb) = 0$$
$$A(\kappa\cos nb - n\sin nb) - B(\kappa\sin nb + n\cos nb) = 0.$$

From these we find

$$\left.\begin{array}{l} (1)\ B=0,\ n\sin nb - \kappa\cos nb = 0,\ C = Ae^{\kappa b}\cos nb = D \\ (2)\ A=0,\ \kappa\sin nb + n\cos nb = 0,\ C = Be^{\kappa b}\cos nb = -D \end{array}\right\} \quad . \quad . \quad . \quad (39)$$

Putting all these together we obtain

$$\left.\begin{array}{ll} \text{inside the plate} & P=\Sigma^{(3)}\cos m\phi J_m(\kappa\rho)(A\cos nze^{-\lambda t}+B\sin n'ze^{-\lambda' t}) \\ & \qquad\qquad +\text{similar terms in } \sin m\phi \\ \text{outside, } z+^{\text{ve}}, & P=\Sigma^{(3)}\cos m\phi J_m(\kappa\rho)(A\cos nbe^{-\lambda t}+B\sin n'be^{-\lambda' t})e^{-\kappa(z-b)}+\,.\,. \\ \text{outside, } z-^{\text{ve}}, & P=\Sigma^{(3)}\cos m\phi J_m(\kappa\rho)(A\cos nbe^{-\lambda t}-B\sin n'be^{-\lambda' t})e^{\kappa(z+b)}+\,.\,. \end{array}\right\} \quad (40)$$

where

$$n\sin nb - \kappa\cos nb = 0,\quad \lambda = \frac{\sigma}{4\pi}(\kappa^2+n^2)$$
$$n'\cos n'b + \kappa\sin n'b = 0,\quad \lambda' = \frac{\sigma}{4\pi}(\kappa^2+n'^2)$$

The summations are to be extended, first over all the values of n and n' corresponding to the roots of the above equations, then over all values of κ from 0 to ∞, and finally over all integral values of m from 0 to ∞; the summation with respect to κ will be of the nature of an integral, as will presently appear.

§ 6. The investigation of the values of the coefficients is attended with some difficulty owing to the difficulty of interpreting the values of J_m for infinite values of the argument. I have therefore sought to evade these difficulties by conceiving the plate as the limit of a spherical shell of finite thickness but of infinite radius, and keeping in view the general course which the solution for a spherical shell takes. It is possible to obtain the solution for any spherical shell, and it might seem therefore easy at once to adapt that solution to the present case: but unfortunately the adaptation is also beset with difficulties of a peculiar kind, and therefore we can do no more than take the steps of that investigation as guides in the present problem. The main light which the case of the spherical shell gives us is that we have to regard $\cos m\phi J_m(\kappa\rho)$ as a degraded form of the spherical surface-harmonic $\cos m\phi P_m^n(\cos\theta)$, obtained by putting

$$\sin\theta = \frac{\rho}{a},\ n = \kappa a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A_1)$$

a being the mean radius of the shell (sensibly constant), κ a finite quantity: to make the passage clear we know that, if $s=\sin\theta$, P_m^n satisfies the equation

$$(1-s^2)\frac{d^2y}{ds^2}+\frac{1-2s^2}{s}\frac{dy}{ds}+\left\{\left(n(n+1)-\frac{m^2}{s^2}\right)\right\}y=0 \quad . \quad . \quad . \quad . \quad . \quad (A_2)$$

(see HEINE, 'Kugelfunctionen,' p. 216, 2nd ed.); which we may satisfy by

$$y=C_m^n(s^m+A_1s^{m+2}+A_2s^{m+4}+\ldots)$$

where

$$A_1\{(m+2)^2-m^2\}+n(n+1)-m.(m+1)=0$$
$$A_2\{(m+4)^2-m^2\}+[n(n+1)-(m+2)(m+3)]A_1=0$$
$$A_3\{(m+6)^2-m^2\}+[n.(n+1)-(m+4)(m+5)]A_2=0, \text{ \&c.}$$

If we now put n (infinitely great) $=\kappa a$, $s=\frac{\rho}{a}$, we find

$$P_m^n=\frac{C_m^n}{a^m}\rho^m\left(1-\frac{\kappa^2\rho^2}{4.\overline{m+1}}+\frac{\kappa^4\rho^4}{4.8.\overline{m+1}.\overline{m+2}}-\frac{\kappa^6\rho^6}{4.8.12.\overline{m+1}.\overline{m+2}.\overline{m+3}}+\ldots\right)$$
$$=\frac{2^m m!}{\kappa^m a^m}C_m^n.J_m(\kappa\rho) \qquad (A_3)$$

the value of the constant C_m^n being obtained from considering the value of $P_m^n \div s^m$ when $s=0$; in fact we have (HEINE, 2nd ed., p. 207),

$$C_m^n=2^{n-m}\frac{n!(n+m)!}{m!(2n)!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A_4)$$

This constant we shall keep, for the present, unreduced.

From the theorem $\int_0^\pi P_m^n.P_m^{n'}\sin\theta d\theta=0$, we derive at once

$$\int_0^\infty J_m(\kappa\rho)J_m(\kappa'\rho)\rho d\rho=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A_5)$$

We have moreover (see HEINE, pp. 327 and 253),

$$\int_0^\pi(P_m^n)^2\sin\theta d\theta=\frac{2}{2n+1}.\frac{(n+m)!(n-m)!}{(1.3.5\ldots\overline{2n-1})^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (A_6)$$

and, correspondingly,

$$\int_0^\infty \overline{J_m(\kappa\rho)}|^2\rho d\rho=\frac{\kappa^{2m}a^{2m+2}}{(C_m^n)^2}.\frac{2}{2n+1}.\frac{(n+m)!(n-m)}{(1.3\,5\ldots2n-1)^2}.\frac{1}{2^{2m}(m!)^2}$$

If we take into account that n is infinitely great compared to m, we see that $(n-m)!=n!\div n^m$ and $(n+m)!=n!\times n^m$; also

$$\frac{1}{C_m^n}=2^m m!\frac{1.3.5\ldots(2n-1)}{n!\,n^m},$$

whence

$$\int_0^\infty [J_m(\kappa\rho)]^2\rho d\rho=\frac{\kappa^{2m}a^{2m+2}}{n^{2m}}\cdot\frac{1}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A_7)$$

But $n=\kappa a$ and the successive values of n correspond to the successive integers $\therefore\ \delta\kappa=\frac{1}{a}$; hence the above integral becomes

$$\int_0^\infty [J_m(\kappa\rho)]^2\rho d\rho=\frac{1}{\kappa\delta\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A_8)$$

We are now in a position to find the values of the coefficients (A), (B) in the expressions (40): for it may be easily shown that, if n_1, n_2 be two different values of n, n'_1, n'_2 two different values of n' from equations (39), then

$$\left.\begin{array}{c}\displaystyle\int_{-b}^{+b}\cos n_1z\cos n_2zdz=0,\quad\int_{-b}^{+b}\cos nz\sin n'zdz=0,\quad\int_{-b}^{+b}\sin n'_1z\sin n'_2zdz=0\\ \displaystyle\int_{-b}^{+b}\cos^2 nzdz=\frac{2nb+\sin 2nb}{2n},\quad\int_{-b}^{+b}\sin^2 n'zdz=\frac{2n'b-\sin 2n'b}{2n'}\end{array}\right\} \quad . \quad (41)$$

From these results we may separate the different values of A, B: for since when $t=0$, $P=-P_0$.

We have in fact

$$A=-\frac{1}{\pi}\cdot\frac{2n}{2nb+\sin 2nb}\cdot\kappa\delta\kappa\cdot\int_0^{2\pi}\cos m\phi d\phi\int_0^\infty\rho J_m(\kappa\rho)d\rho\int_{-b}^{+b}P_0\cos nzdz \quad . \quad . \quad (42)$$

B may be found in a similar manner, and likewise also the terms depending on sin $m\phi$. Collecting all these results for the value of P (ϕ, ρ, z) at an external point on the positive side of the plate, we have (z positive)

$$P=-\frac{1}{\pi}\Sigma_m\int_0^{2\pi}\cos m(\phi-\phi')d\phi'\int_0^\infty\kappa e^{-\kappa(z-b)}J_m(\kappa\rho)d\kappa\int_0^\infty\rho' J_m(\kappa\rho')d\rho'\Sigma$$

$$\left\{e^{-\lambda t}\frac{2n\cos nb}{2nb+\sin 2nb}\int_{-b}^{+b}P'_0\cos nz'dz'+e^{-\lambda' t}\cdot\frac{2n'\sin n'b}{2n'b-\sin 2n'b}\int_{-b}^{+b}P'_0\sin n'z'dz'\right\} \quad . \quad . \quad (43)$$

The last Σ is to be extended over all the values of n, n' which can be derived as roots of the equations in these quantities. The first Σ denotes that the summation is to be extended to all values of m. It is to be feared there is no simpler mode of writing the above result; but the approximate solution of any given case could be obtained from it by carefully conducting the approximations.

Case of an indefinitely thin plane at rest.

§ 7. We may pass from the above problem to the present by making b indefinitely small. If R be the resistance to the current per unit of area

$$2\mathrm{R}b=\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

and we have also to enquire what the values of n and n' derived—

(*a.*) From the former solution will now become. To do so I shall write the equation for n in the form

$$\theta \sin \theta-\kappa b \cos \theta=0, \text{ where } \theta=nb.$$

(1.) When κb is small the roots of the equation are approximated to, *in general*, by $\sin \theta=0$, or $\theta=j\pi=\beta$, say.

Let $\theta=\beta+x$, then to find x we have

$$(\beta+x) \sin x=\kappa b \cos x.$$

Expanding $\sin x$ and $\cos x$ in ascending powers of x, we obtain, after some reduction, x in the following series of powers of κb

$$x=\frac{1}{\beta}.\kappa b-\frac{1}{\beta^3}\kappa^2 b^2-\frac{6+\beta^2}{5\beta^5}\kappa^3 b^3+\ldots$$

moreover

$$\lambda=\frac{\mathrm{R}b}{2\pi}(\kappa^2+n^2).$$

Hence, *in general*,

$$\left.\begin{aligned} nb&=j\pi+\frac{1}{j\pi}\kappa b-\frac{1}{j^3\pi^3}\kappa^2 b^2-\frac{6+j^2\pi^2}{3j^5\pi^5}\kappa^3 b^3+\ldots \\ \lambda&=\frac{\mathrm{R}}{2\pi b}\left(j^2\pi^2+2\kappa b+\frac{j^2\pi^2-1}{j^2\pi^2}\kappa^2 b^2-\frac{18+2j^2\pi^2}{3j^4\pi^4}.\kappa^3 b^3+\ldots\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (45)$$

(2.) But there is a root of the equation for which $n\ b$ is very small given approximately by $n^2b^2=\kappa b$; expanding fully the equation $\theta \sin \theta-\kappa b \cos \theta=0$ in ascending powers of θ, we find

$$\left.\begin{aligned} n^2 b^2&=\kappa b-\tfrac{1}{3}\kappa^2 b^2+\tfrac{4}{45}\kappa^3 b^3+\ldots \\ \lambda&=\frac{\mathrm{R}}{2\pi}(\kappa+\tfrac{2}{3}\kappa^2 b+\tfrac{4}{45}\kappa^3 b^2+\ldots) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

(*b.*) In like manner, the equation in n', which is

$$n'b \cos n'b + \kappa b \sin n'b = 0,$$

is satisfied, approximately, by $n'b = \frac{2i+1}{2} . \pi = \alpha$, say: and more accurately by $n'b = \alpha + x$, where

$$(\alpha + x) \sin x = \kappa b \cos x.$$

This is precisely the same equation as before, and gives

$$\left.\begin{aligned} n'b &= \alpha + \frac{1}{\alpha} . \kappa b - \frac{1}{\alpha^3}\kappa^2 b^2 - \frac{6+\alpha^2}{3\alpha^5}\kappa^3 b^3 \\ \lambda' &= \frac{R}{2\pi b}\left(\alpha^2 + 2\kappa b + \frac{\alpha^2-1}{\alpha^2}\kappa^2 b^2 - \frac{18+2\alpha^2}{3\alpha^4}\kappa^3 b^3 + \ldots\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (47)$$

When we make b indefinitely small, retaining R as a finite quantity, all the values of λ and λ' become indefinitely great except those which arise from the approximation in n, $n^2 b^2 = \kappa b + \ldots$; λ is then $= \frac{R\kappa}{2\pi}$. Rejecting therefore all other values of n, n', λ and λ', and confining our attention to the parts of P outside the plate, we find [writing A cos nb under the single term $\mathfrak{A}$],

$$\left.\begin{aligned} &z \text{ positive,} \quad P = \Sigma \cos m\phi . J^m(\kappa\rho) . \mathfrak{A} e^{-\kappa\left(z + \frac{Rt}{2\pi}\right)} \\ &z \text{ negative,} \quad P = \Sigma \cos m\phi . J^m(\kappa\rho) . \mathfrak{A} e^{+\kappa\left(z - \frac{Rt}{2\pi}\right)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

the term in b in the exponential e being neglected.

If, at the surface of the plate, $P = f(x, y, t)$, then at any point for which z is positive,

$$\left.\begin{aligned} &P = f\left(x, y, t + \frac{2\pi z}{R}\right) \\ \text{and at any point, } z \text{ negative,} \quad &P = f\left(x, y, t - \frac{2\pi z}{R}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

This is MAXWELL's result.

I have dwelt somewhat fully upon this case because it brings out very clearly the mode in which the terms depending upon all but a certain number of the values of λ and λ' disappear when b is made indefinitely small. When we consider the case of a spherical shell, we shall find that a similar reduction takes place.

Induction of currents in a solid sphere at rest.

§ 8. When a solid sphere of radius a is influenced by an external magnetic system whose potential is Ω_0, we may express the vector potential of the latter by taking

$$\Omega_0=\frac{d}{dr}(P_0r),\ F_0=0,\ G_0=-\frac{dP_0}{\sin\theta d\phi},\ H_0=\frac{dP_0}{d\theta}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{50}$$

dx, dy, dz being now coincident with *dr, rdθ, r* sin *θdφ*.

The equations of electromotive force, viz.:

$$\left.\begin{aligned}\sigma u&=-\frac{d(F+F_0)}{dt}-\frac{d\psi}{dr}\\ \sigma v&=-\frac{d(G+G_0)}{dt}-\frac{d\psi}{rd\theta}\\ \sigma w&=-\frac{d(H+H_0)}{dt}-\frac{d\psi}{r\sin\theta d\phi}\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{51}$$

and all the conditions of the problem may be satisfied by taking

$$\left.\begin{aligned}&\psi=0,\ F=0,\ G=-\frac{dP}{\sin\theta d\phi},\ H=\frac{dP}{d\theta}\\ &u=0,\ v=-\frac{d\Phi}{\sin\theta d\phi},\ w=\frac{d\Phi}{d\theta}\\ &4\pi\Phi=-\nabla^2P.\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{52}$$

P satisfies the characteristic equation

$$\frac{\sigma}{4\pi}\nabla^2P=\frac{dP}{dt},$$

or, more fully,

$$\frac{d^2P}{dr^2}+\frac{2}{r}\frac{dP}{dr}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{d^2P}{d\phi^2}=\frac{4\pi}{\sigma}\frac{dP}{dt};\quad . \quad . \quad . \quad . \tag{53}$$

outside the sphere P satisfies the equation

$$\nabla^2P=0;$$

and we must have the further condition that F, G, H and their differential coefficients, or (which is the same thing)

$$P\text{ and }\frac{dP}{dr}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{54}$$

are to undergo no sudden change in passing across the surface of the sphere, and that P must be finite at the centre.

The equation in P (inside the sphere), is satisfied by

$$P=AU_nS_n(\lambda r)e^{-\lambda^2\frac{\sigma t}{4\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55)$$

where U_n is any surface harmonic of degree n, and $S_n(\lambda r)$ is the solution of the equation

$$\frac{d^2Q}{dr^2}+\frac{2}{r}\frac{dQ}{dr}+\left(\lambda^2-\frac{n(n+1)}{r^2}\right)Q=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (B)$$

which does not become infinite at the centre. In fact the two solutions of this equation are, as I have elsewhere shown

$$\left.\begin{aligned} S_n(\lambda r)&=(\lambda r)^n.\left(\frac{1}{\lambda r}\frac{d}{d\lambda r}\right)^n.\frac{\sin\lambda r}{\lambda r} \\ T_n(\lambda r)&=(\lambda r)^n\left(\frac{1}{\lambda r}\frac{d}{d\lambda r}\right)^n\frac{\cos\lambda r}{\lambda r} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (B_1)$$

of which the former vanishes at the centre, except when $n=0$, and the latter vanishes at ∞; it is the former of these expressions which is chosen.

Outside the sphere the value of P is

$$P=BU_n.r^{-n-1}e^{-\lambda^2\frac{\sigma t}{4\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56)$$

and the above boundary conditions give

$$A_nS_n(\lambda a)=B_na^{-n-1},\ A_n\frac{dS_n}{da}=-B_n(n+1)a^{-n-2},$$

or

$$B=a^{n+1}.A_nS(\lambda a)$$

$$a\frac{dS_n}{da}+(n+1)S_n=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (57)$$

This equation is also equivalent to the following somewhat simpler one (see equation B_5, § 10),

$$S_{n-1}(\lambda a)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (57_1)$$

The values of λ being thus determined, we may proceed as in the former problem to find the coefficients due to any initial circumstances, and thence the state of the sphere may be readily deduced by the principles which have been already explained. I may observe that U_n is to be expressed in the system of tesseral harmonics

$$\sum_0^n(A\cos m\phi+B_m\sin m\phi)P_m^n$$

The rest of the process needs no remark.

Spherical shell of finite thickness.

§ 9. Let the external and internal radii be b, a; all the conditions of the problem will be satisfied by the same general assumptions as to the forms of the electromagnetic momentum of the currents, the current function and of the potentials of the external magnetism. The characteristic equation remains

$$\frac{\sigma}{4\pi}\nabla^2 P=\frac{dP}{dt};$$

and, as in the former cases, since there is nowhere flow along the radii to the common centre of the surfaces and no free electricity, the currents are entirely confined to the interior of the shell, and we shall have as before for boundary conditions, that the vector potential of the currents in the shell and its differential coefficients sustain no sudden change at crossing the surfaces. Outside the shell and within its inner surface, the vector potential is given by expressions of the same form as in the substance, viz.:

$$F=0,\quad G=-\frac{dP}{\sin\theta d\phi},\quad H=\frac{dP}{d\theta},$$

but in the space free from currents

$$\nabla^2 P=0.$$

At the boundaries all the conditions are satisfied by having

$$P \text{ and } \frac{dP}{dr}$$

continuous when $r=a$ and when $r=b$; we must therefore take

$$\left.\begin{array}{ll}\text{within sphere of radius a,} & P=\Sigma Cr^nU_ne^{-\lambda^2\frac{\sigma}{4\pi}.t}\\ \text{in substance of shell,} & P=\Sigma.(AS_n+BT_n)U_ne^{-\lambda^2.\frac{\sigma t}{4\pi}}\\ \text{without sphere of radius b,} & P=\Sigma Dr^{-n-1}.U_ne^{-\lambda^2.\frac{\sigma t}{4\pi}}\end{array}\right\} \quad . \quad . \quad . \quad . \quad (58)$$

S_n and T_n are the expressions referred to in last article: they are the particular solutions of the equation

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\lambda^2-\frac{n(n+1)}{r^2}\right)R=0.$$

If we write $x=\lambda r$, S_n and T_n are respectively equal to

$$x^n.\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x} \text{ and } x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n.\frac{\cos x}{x}.$$

To express the boundary conditions we shall write

$$\alpha=\lambda a,\ \beta=\lambda b,\ S=S(\lambda a),\ S'=S(\lambda b),\ \&c\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (59)$$

we have, accordingly,

$$\left.\begin{aligned} &Ca^n=AS+BT,\ nCa^{n-1}=\lambda\left(A\frac{dS}{d\alpha}+B\frac{dT}{d\alpha}\right) \\ &Db^{-n-1}=AS'+BT',-(n+1)Db^{-n+2}=\lambda\left(A\frac{dS'}{d\beta}+B\frac{dT'}{d\beta}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (60)$$

These equations may be put into much more elegant forms; but to do so we require various properties of the S— and T— functions; and as these are not very generally understood, I shall here digress into a brief sketch of the whole subject, confining myself as strictly as possible to those properties which have a direct bearing on the present question.

[*Properties of the "Spherical Functions."*

§ 10. If we write $x^2=t$, and work out the differentiations and integrations, it can be readily shown that

$$\left.\begin{aligned} x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x}&=(-1)^n\ x^{-n-1}.\left(\frac{1}{x}\frac{d}{dx}\right)^{-n-1}\frac{\cos x}{x} \\ x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\cos x}{x}&=(-1)^{n+1}x^{-n-1}\left(\frac{1}{x}\frac{d}{dx}\right)^{-n-1}\frac{\sin x}{x} \end{aligned}\right\} \quad . \quad . \quad . \quad . \qquad (B_3)$$

The constants introduced by integration must be so adjusted as to make a certain number of terms in the second equality coincide: the others will be exactly equal.

If we write each member of these two equations, as before, respectively S_n and T_n, and differentiate the quantities on the left-hand sides of the equations, we find

$$xS_{n+1}=x\frac{dS_n}{dx}-nS_n,\ xT_{n+1}=x\frac{dT_n}{dx}-nT_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (B)$$

Differentiating the right-hand members we obtain

$$-xS_{n-1}=x\frac{dS_n}{dx}+(n+1)S_n,\quad -xT_{n-1}=x\frac{dT_n}{dx}+(n+1)T_n \quad . \quad . \quad . \quad . \quad (B_5)$$

These are the fundamental formulæ of reduction in these functions: if in the latter we write $n+1$ for n, and substitute the values of S_{n+1} and T_{n+1} from the former, we can show easily that S_n and T_n satisfy the differential equation

$$\frac{d^2R}{dx^2}+\frac{2}{x}\frac{dR}{dx}+\left(1-\frac{n.n+1}{x^2}\right)R=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

and are therefore the two solutions of it. If we eliminate from the above $x\frac{dS_n}{dx}$, and $x\frac{dT_n}{dx}$, we find

$$\left.\begin{aligned}(2n+1)\frac{S_n}{x}+S_{n+1}+S_{n-1}=0\\(2n+1)\frac{T_n}{x}+T_{n+1}+T_{n-1}=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_6)$$

whence also

$$S_{n+1}T_n-T_{n+1}S_n=S_nT_{n-1}-S_{n-1}T_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_7)$$

and, by successive reductions,

$$S_{n+1}T_n-T_{n+1}S_n=S_1T_0-T_1S_0=\frac{1}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_8)$$

and thus also, by (B_6),

$$S_{n+1}T_{n-1}-T_{n+1}S_{n-1}=-\frac{2n+1}{x^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_9)$$

We obtain also directly from equations (B_4),

$$T_n\frac{dS_n}{dx}-S_n\frac{dT_n}{dx}=S_{n+1}T_n-T_{n+1}S_n=\frac{1}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_{10})$$

a well-known result.]

11. If we now turn to equations (60), we find by elimination

$$\begin{aligned}\lambda A\left(T\frac{dS}{d\alpha}-S\frac{dT}{d\alpha}\right)&=-a^{n-1}C\left(\alpha\frac{dT}{d\alpha}-nT\right)\\\lambda B\left(T\frac{dS}{d\alpha}-S\frac{dT}{d\alpha}\right)&=+a^{n-1}C\left(\alpha\frac{dS}{d\alpha}-nS\right)\\\lambda A\left(T'\frac{dS'}{d\beta}-S'\frac{dT'}{d\beta}\right)&=-b^{-n-2}D\left(\beta\frac{dT'}{d\beta}+(n+1)T'\right)\\\lambda B\left(T'\frac{dS'}{d\beta}-S'\frac{dT'}{d\beta}\right)&=+b^{-n-2}D\left(\beta\frac{dS'}{d\beta}+(n+1)S'\right)\end{aligned}$$

These may be written

$$\mathrm{A}=-\lambda^2\mathrm{a}^{n+2}\mathrm{T}_{n+1}(\lambda\mathrm{a})\mathrm{C},\qquad \mathrm{B}=+\lambda^2\mathrm{a}^{n+2}\mathrm{S}_{n+1}(\lambda\mathrm{a}).\mathrm{C}$$

$$\mathrm{A}=+\lambda^2\mathrm{b}^{-n+1}\mathrm{T}_{n-1}(\lambda\mathrm{b}).\mathrm{D},\quad \mathrm{B}=-\lambda^2\mathrm{b}^{-n+1}\mathrm{S}_{n-1}(\lambda\mathrm{b}).\mathrm{D}.$$

Introducing a new constant $\mathfrak{A}$, we may therefore put

$$\left.\begin{aligned}\mathrm{C}&=\quad\mathfrak{A}.\mathrm{b}^{-n+1}\mathrm{S}_{n-1}(\lambda\mathrm{b})\\ \mathrm{A}&=-\mathfrak{A}\lambda^2\mathrm{a}^{n+2}\mathrm{b}^{-n+1}\mathrm{T}_{n+1}(\lambda\mathrm{a})\mathrm{S}_{n-1}(\lambda\mathrm{b})\\ \mathrm{B}&=+\mathfrak{A}\lambda^2\mathrm{a}^{n+2}\mathrm{b}^{-n+1}\mathrm{S}_{n+1}(\lambda\mathrm{a})\mathrm{S}_{n-1}(\lambda\mathrm{b})\\ \mathrm{D}&=-\mathfrak{A}\mathrm{a}^{n+2}\mathrm{S}_{n+1}(\lambda\mathrm{a})\end{aligned}\right\}\quad\ldots\ldots\quad(61)$$

with the further equation, for the determination of λ,

$$\mathrm{S}_{n+1}(\lambda\mathrm{a}).\mathrm{T}_{n-1}(\lambda\mathrm{b})-\mathrm{T}_{n+1}(\lambda\mathrm{a}).\mathrm{S}_{n-1}(\lambda\mathrm{b})=0\quad\ldots\ldots\quad(62)$$

To construct this equation, I observe that

$$\mathrm{S}_n,\ =x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x},$$

may be written

$$\mathrm{X}_1{}^n\sin\left(x+\frac{n\pi}{2}\right)+\mathrm{X}_2{}^n\sin\left(x+\frac{n+1}{2}\pi\right)\quad\ldots\ldots\quad(\mathrm{B}_{11})$$

and that

$$\mathrm{T}_n,\ \text{which}\ =x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin\left(\frac{\pi}{2}+x\right)}{x},$$

may in like manner be written

$$\mathrm{X}_1{}^n\sin\left(x+\frac{n+1}{2}\pi\right)+\mathrm{X}_2{}^n\sin\left(x+\frac{n+2}{2}\pi\right)\quad\ldots\ldots\quad(\mathrm{B}_{12})$$

In fact

$$\left.\begin{aligned}x.\mathrm{X}_1{}^n&=1-\frac{(n-1)n(n+1)(n+2)}{2.4}.\frac{1}{x^2}+\frac{(n-3)(n-2)\ \ldots\ (n+3)(n+4)}{2.4.6.8}.\frac{1}{x^4}-\&\text{c.}\\ x.\mathrm{X}_2{}^n&=\frac{n(n+1)}{2}.\frac{1}{x}-\frac{(n-2)(n-1)\ldots(n+2)(n+3)}{2.4.6}.\frac{1}{x^3}+\ldots\end{aligned}\right\}\quad(\mathrm{B}_{13})$$

With this notation we may put

$$S_{n+1}(\lambda a)=A_1^{n+1}\sin\left(\lambda a+\frac{n+1}{2}\pi\right)+A_2^{n+1}\cos\left(\lambda a+\frac{n+1}{2}\pi\right)$$

$$T_{n+1}(\lambda a)=A_1^{n+1}\cos\left(\lambda a+\frac{n+1}{2}\pi\right)-A_2^{n+1}\sin\left(\lambda a+\frac{n+1}{2}\pi\right),$$

$$S_{n-1}(\lambda b)=B_1^{n-1}\sin\left(\lambda b+\frac{n-1}{2}\pi\right)+B_2^{n-1}\cos\left(\lambda b+\frac{n-1}{2}\pi\right)$$

$$T_{n-1}(\lambda b)=B_1^{n-1}\cos\left(\lambda b+\frac{n-1}{2}\pi\right)-B_2^{n-1}\sin\left(\lambda b+\frac{n-1}{2}\pi\right)$$

Substituting in (62),

$$(A_2^{n+1}B_1^{n-1}-A_1^{n+1}B_2^{n-1})\cos(\lambda(b-a)-\pi)-(A_2^{n+1}B_2^{n-1}+A_1^{n+1}B_1^{n-1})\sin(\lambda(b-a)-\pi)=0$$

or more simply

$$(A_1^{n+1}B_1^{n-1}+A_2^{n+1}B_2^{n-1})\sin.\lambda(b-a)-(A_2^{n+1}B_1^{n-1}-A_1^{n+1}B_2^{n-1})\cos.\lambda(b-a)=0.$$

This equation may be further simplified; for, putting

$$\tan\vartheta'=\frac{X_2^n}{X_1^n},\ \tan\alpha'=\frac{A_2^{n+1}}{A_1^{n+1}},\ \tan\beta'=\frac{B_2^{n-1}}{B_1^{n-1}}\quad\ldots\ldots\quad(63)$$

it becomes

$$\sin\{\lambda.\overline{b-a}-\overline{\alpha'-\beta'}\}=0,$$

whence

$$\lambda.\overline{b-a}=\alpha'-\beta'+i\pi\quad\ldots\ldots\ldots\ldots\quad(64)$$

here α' and β' are the smallest positive angles which satisfy the above equations; and ϑ may, when $2x>n(n+1)$, be expanded in ascending powers of $\frac{1}{x}$ by GREGORY'S series.

Up to $\frac{1}{x^3}$,

$$\vartheta=\frac{n(n+1)}{2x}+\frac{(n-1)n(n+1)(n+2)}{2.4}.\frac{1}{x^2}-\frac{n(n+1)}{2.4.6}[3.(n^2+n)^2-8(n^2+n)+12].\frac{1}{x^3}+\ldots\quad(65)$$

The values of α' and β' in terms of λa and λb may be similarly found, and when substituted in equation (64) will give the means of finding all the values of λ. These being found, the complete expression for P may be written down and the values of the constants therein determined from the initial circumstances by known methods; we shall therefore proceed to find the value of P at a point outside the shell.

The potentials at a point outside the shell and in its substance may be written respectively

$$P=\Sigma D'\left(\frac{b}{r}\right)^{n+1}U_n e^{-\frac{\lambda^2\sigma}{4\pi}t}$$

and

$$P=\Sigma D'\lambda^2 b^2(S_n(x)T_{,-1}(\beta)-T_n(x)S_{,-1}(\beta))e^{-\frac{\lambda^2\sigma}{4\pi}t},$$

wherein, as before, $x=\lambda r$, $\beta=\lambda b$.

Let us write

$$H_n=(S_nT_{n-1}(\beta)-T_nS_{n-1}(\beta))\lambda^2 b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

where $T_{n-1}(\beta)$ and $S_{n-1}(\beta)$ are to be treated as simply certain constant multipliers; H_n will satisfy all the formulæ of reduction which we have found to hold separately for S_n and T_n; and we have further these particular values

$$\left.\begin{array}{l} H_n(\beta)=1,\ H_{n+1}(\alpha)=0 \text{ by equation (62)} \\ \text{and by the equation } (B_0),\ H_{n+1}(\beta)=-\frac{2n+1}{\beta}H_n(\beta)=-\frac{2n+1}{\lambda b} \end{array}\right\} \quad . \quad . \quad (67)$$

Now $H_n(x)$ satisfies the equation

$$\frac{d^2H}{dr^2}+\frac{2}{r}\frac{dH}{dr}+\left(\lambda^2-\frac{n(n+1)}{r^2}\right)H=0,$$

and if H' correspond to any other value of λ, viz.: λ', we have also

$$\frac{d^2H'}{dr^2}+\frac{2}{r}\frac{dH'}{dr}+\left(\lambda'^2-\frac{n(n+1)}{r^2}\right)H'=0,$$

whence

$$\left\{r^2\left(H\frac{dH'}{dr}-H'\frac{dH}{dr}\right)\right\}_a^b+(\lambda'^2-\lambda^2)\int_a^b r^2HH'dr=0.$$

But, by equations (B_4),

$$r\frac{dH_n}{dr}=nH_n+\lambda rH_{n+1}$$

$$r\frac{dH'_n}{dr}=nH'_n+\lambda' rH'_{n+1}$$

and, therefore,

$$(\lambda'^2-\lambda^2)\int_a^b r^2HH'dr=\left\{r^3(\lambda H_{n+1}H'_n-\lambda' H_nH'_{n+1})\right\}_a^b \quad . \quad . \quad . \quad . \quad . \quad (68)$$

But if λ' be also one of the roots of equation (62), both H_{n+1} and H'_{n+1} vanish at the lower limit; and at the upper limit

$$H_n = H'_n = 1 \text{ and } \lambda H_{n+1} = \lambda' H'_{n+1} = -\frac{2n+1}{b}$$

hence

$$\int_a^b r^2 H_n H'_n dr = 0 \text{ . . . . } \quad (69)$$

If we put $\lambda' = \lambda + \delta\lambda$, and then make $\delta\lambda$ infinitely small, we arrive at the value of $\int_a^b r^2 H^2 dr$: for

$$H'_n = H_n(\lambda r + \delta\lambda . r) = H_n + r\delta\lambda\frac{dH_n}{dx}$$

$$= H_n + \frac{1}{\lambda}\delta\lambda(\lambda r H_{n+1} + nH_n) \text{ by } (B_4)$$

and

$$H'_{n+1} = H_{n+1}(\lambda r + \delta\lambda . r) = H_{n+1} + \frac{\delta\lambda}{\lambda}(-\lambda r H_n - \overline{n+1}H_{n+1}) \text{ by } (B_5)$$

Putting in these values in (68)

$$\int_a^b r^2 H_n^2 dr = \frac{1}{2\lambda}\Big[r^3\{\lambda r(H_n^2 + H^2_{n+1}) + 2nH_n H_{n+1}\}\Big]_a^b$$

$$= \frac{b^2}{2\lambda^2}(2n+1) - \frac{\lambda^2 b^2 a^4}{2}(S_n(\lambda a)T_{n-1}(\lambda b) - T_n(\lambda a)S_{n-1}(\lambda b))^2 \quad . \quad (70)$$

Let us call this integral I_n; we may now separate the coefficient D′ due to any term of U_n, say $\cos m\phi P_m^n$: for if we put in equation (58) $t=0$, $P=-P_0$, multiply across by $\cos m\phi d\phi . P_m^n \sin\theta d\theta . H_n r^2 dr$ and integrate throughout the body of the shell, all the terms on the right hand side disappear except those which have the some *m*, *n*, and λ; and, if we note that

$$\int_0^{2\pi}\cos^2 m\phi d\phi = \pi, \ \int_0^{\pi}(P_m^n)^2 \sin\theta d\theta = \frac{2}{2n+1} \cdot \frac{(n+m)!\,(n-m)!}{(1.3.5\ldots\overline{2n-1})^2} = J_m^n \text{ say,}$$

we find

$$D' = -\int_0^{2\pi}\cos m\phi d\phi \int_0^{\pi} P_m^n . \sin\theta d\theta \int_a^b P_0 H_n r^2 dr \div \pi J_m^n I_n.$$

The coefficient due to $\sin m\phi P_m^n$ may be similarly found.

Putting all the results together we find for the value of P at an outside point

$$\mathrm{P}=-\frac{1}{\pi}\,{}_m\sum_0^\infty\,{}_n\sum_0^\infty\left(\frac{\mathrm{b}}{r}\right)^{n+1}{}_\lambda\Sigma\int_0^{2\pi}\cos m(\phi-\phi')d\phi'\,.\int_0^\pi \mathrm{P}_m{}^n(\theta)\mathrm{P}_m{}^n(\theta')\sin\theta' d\theta'\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{H}_n(\lambda r')\frac{\mathrm{P}'_0 r'^2}{\mathrm{I}_n\mathrm{J}_m{}^n}dr'$$

$$\times e^{-\frac{\lambda^2\sigma}{4\pi}t}\quad .\quad .\quad .\quad .\quad .\quad (71)$$

From this formula the action of the shell on an external point due to any inducing system may be found.

We proceed to inquire what will be the result when the shell is supposed to become infinitely thin.

Infinitely thin shell.

§ 13. Putting $\mathrm{b}-\mathrm{a}=c$, $\sigma=\mathrm{R}c$, the exponential-function $e^{-\lambda^2 c\frac{\mathrm{R}t}{4\pi}}$, will be finite only for such values of λ as make $\lambda^2 c$ finite: if $\lambda^2 c$ become indefinitely great the corresponding terms will rapidly disappear. More accurately, the terms for which $\lambda^2 c$ is finite will die away infinitely less rapidly than those for which it is infinitely great.

If, in the equation

$$\lambda c=\alpha'-\beta'+i\pi,$$

we take as a first approximation $\lambda=\frac{i\pi}{c}$, we may readily expand λc in a series of the form $i\pi+\mathrm{E}_1.c+\mathrm{E}_2c^2+\ldots$; but in this case $\lambda^2 c=\frac{i^2\pi^2}{c}$ approximately when c is very great.

The corresponding system of currents will rapidly decay. It is therefore by putting $i=0$ that we obtain the currents which have the greatest persistence. The equation in λ is

$$\lambda^2 c=\frac{(n+2)(n+1)}{2\mathrm{a}}-\frac{(n-1)n}{2\mathrm{b}}+\text{ terms in }\lambda^{-1},\ \lambda^{-2},\ \ldots,$$

and, for a first approximation, $\mathrm{b}=\mathrm{a}$ and

$$\lambda^2 c=\frac{1}{\mathrm{a}}(2n+1)\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad (72)$$

Hence, by equations (58) and

$$\left.\begin{array}{ll}\text{for points outside the shell,} & \mathrm{P}=\Sigma\mathfrak{C}\left(\frac{\mathrm{a}}{r}\right)^{n+1}\mathrm{U}_n e^{-\frac{\mathrm{R}}{4\pi\mathrm{a}}(2n+1)t}\\ \text{,,}\qquad\text{within the shell,} & \mathrm{P}=\Sigma\mathfrak{C}\left(\frac{r}{\mathrm{a}}\right)^{n}\mathrm{U}_n e^{-\frac{\mathrm{R}}{4\pi\mathrm{a}}(2n+1)t}\end{array}\right\}\quad .\quad .\quad .\quad (73)$$

To interpret these results:—

(1.) For an external point,

$$\left.\begin{aligned}&\text{we observe that when } t=0,\ \mathrm{P}=\Sigma\mathfrak{C}\left(\frac{\mathrm{a}}{r}\right)^{n+1}\mathrm{U}_n\\ &\text{and that, always, } \mathrm{P}=e^{-\frac{\mathrm{R}}{4\pi\mathrm{a}}t}.\Sigma\mathfrak{C}\left(\frac{\mathrm{a}'}{r}\right)^{n+1}\mathrm{U}_n\\ &\text{where } \mathrm{a}'=\mathrm{a}e^{-\frac{\mathrm{R}}{2\pi\mathrm{a}}t}\end{aligned}\right\}\quad\cdots\cdots\quad(74)$$

The result, therefore, of an impulse on the sheet is such that initially the currents exert the same action on an external point, as a positive image of the magnetic system placed at the position of the electric image at the surface of the sheet.

The points in which this imaginary magnetism is distributed then move towards the centre according to the law $\rho'=\rho e^{-\frac{\mathrm{R}t}{2\pi\mathrm{a}}}$, while at the same time the intensity at each point increases according to the law $\mathrm{I}'=\mathrm{I}e^{\frac{\mathrm{R}t}{4\pi\mathrm{a}}}$.

When $\mathrm{a}=\infty$ we may take $\rho=\mathrm{a}$ and $\rho-\rho'=\frac{\mathrm{R}}{2\pi}t$, and I′ is constant; this result reproduces MAXWELL's expressions for a plane sheet.

(2.) For a point on the other side of the shell

$$\left.\begin{aligned}&\mathrm{P}=e^{-\frac{\mathrm{R}t}{4\pi\mathrm{a}}}\Sigma.\mathfrak{C}\left(\frac{r}{\mathrm{a}''}\right)^{n}\mathrm{U}_n\\ \text{where}&\\ &\mathrm{a}''=\mathrm{a}e^{+\frac{\mathrm{R}t}{2\pi\mathrm{a}}}\end{aligned}\right\}\quad\cdots\cdots\quad(75)$$

The effect is, therefore, the same as if the inducing magnetic system were reversed in sign, and the points in which it is distributed were to move off to infinity in lines passing through the centre of the shell according to the law

$$\rho'=\rho e^{\frac{\mathrm{R}t}{2\pi\mathrm{a}}},$$

the intensity at each point diminishing according to the law

$$\mathrm{I}'=\mathrm{I}e^{-\frac{\mathrm{R}t}{4\pi\mathrm{a}}}.$$

§ 14. This result is so interesting that it may be well to give an independent demonstration of it. Employing, as hitherto, spherical coordinates, let Φ be the current function for the currents in the sheet; P, the potential due to imaginary matter distributed over it, with surface density Φ; the magnetic and vector potentials of the current system may be written

$$\Omega=-\frac{1}{\mathrm{a}}\frac{d}{dr}(\mathrm{P}r),\quad \mathrm{F}=0,\quad \mathrm{G}=+\frac{1}{\mathrm{a}}\frac{d\mathrm{P}}{\sin\theta d\phi},\quad \mathrm{H}=-\frac{d\mathrm{P}}{\mathrm{a}d\theta}.$$

The components of the current are

$$u=0,\ v=-\frac{d\Phi}{\sin\theta d\phi},\ w=\frac{d\Phi}{d\theta}.$$

The equations, therefore, of the current are

$$-\frac{R}{4\pi}\frac{d\Phi}{\sin\theta d\phi}=-\frac{d^2P}{a\sin\theta d\phi dt}-\frac{d\psi}{ad\theta}$$

$$+\frac{R}{4\pi}\frac{d\Phi}{d\theta}=+\frac{d^2P}{ad\theta dt}-\frac{d\psi}{a\sin\theta d\phi}.$$

which is satisfied by

$$\psi=0,\ \Phi=\frac{1}{a}\frac{dP}{dt}.$$

Outside the sheet and within its inner surface,

$$F=0,\ G=\frac{dP}{\sin\theta d\phi},\ H=-\frac{dP}{d\theta},$$

$$\nabla^2P=0.$$

Hence

outside $\quad P_0=\Sigma A\left(\frac{a}{r}\right)^{n+1}U_n e^{-\lambda^2\frac{R}{4\pi}t}$

inside $\quad P_\iota=\Sigma A\left(\frac{r}{a}\right)^{n}U_n e^{-\lambda^2\frac{R}{4\pi}t}$;

at the surface $\quad \Phi=-\frac{1}{a}\Sigma AU_n e^{-\lambda^2\frac{R}{4\pi}t}.\frac{\lambda^2R}{4\pi}.$

But we have also the further condition

$$\left(\frac{dP_0}{dr}-\frac{dP_\iota}{dr}\right)_{r=a}=4\pi\Phi,$$

hence

$$\lambda^2=\frac{1}{a}(2n+1);$$

and therefore

$$P_0=\Sigma A\left(\frac{a}{r}\right)^{n+1}U_n e^{-(2n+1)\frac{R}{4\pi a}t},$$

$$P_\iota=\Sigma A\left(\frac{r}{a}\right)^{n}U_n e^{-(2n+1)\frac{R}{4\pi a}t}.$$

This corresponds with the result obtained from the more general investigation of last article.

Moving conductor.

§ 15. When the conductor is moving in any manner, the parts of the electromotive force due to the motion are, in the directions of the three axes, respectively equal to

$$\gamma\dot{y}-\beta\dot{z},\ \alpha\dot{z}-\gamma\dot{x},\ \beta\dot{x}-\alpha\dot{y}.$$

The form of these expressions shows that, when it is desirable to resolve the electromotive force in any other directions at right angles, the components will be

$$\gamma\mathrm{V}-\beta\mathrm{W},\ \alpha\mathrm{W}-\gamma\mathrm{U},\ \beta\mathrm{U}-\alpha\mathrm{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

α, β, γ being the components of magnetic force, and U, V, W the components of velocity in these directions. When the motion of the body is uniform, as when it is revolving uniformly round an axis, and when it is symmetrical about this axis, the electric state may after a certain time be supposed to have become constant; there is then no variation of electric displacement at each point of space, and the currents of conduction become the total currents. We have then

$$u=\sigma\mathrm{P}=\frac{\sigma}{\mathrm{K}}\cdot f, \, . \, .$$

and therefore

$$\epsilon=\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

There is thus no free electricity in the substance of the conductor, though there may be electric potential: and the normal component of the current across the surface of the body is zero: that is to say,

$$lu+mv+nw=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is to be noted here that we are here dealing with the state of definite points of space: these are invariable. The different parts of the conductor take different conditions as they move from one point of space to another.

Since the currents are confined to the conductor the vector potential due to them and its differential coefficients will all be continuous on passing across the surface of the conductor.

With these preliminary observations we proceed to the consideration of a solid of infinite extent and thickness bounded by a plane face, revolving round an axis normal to its face.

Infinite solid bounded by plane face.

§ 16. Let the axis of z be chosen to coincide with the axis of revolution and the origin on the face, and revolve along $d\rho$, $\rho d\phi$, dz;

$$\mathrm{U}=0,\ \mathrm{V}=\omega\rho,\ \mathrm{W}=0 \quad \ldots\ldots\ldots\ldots \quad (4)$$

ω being the angular velocity supposed uniform.

The components of electromotive force are

$$\left.\begin{aligned}\mathrm{P}&=\ \ \omega\rho\gamma-\frac{d\psi}{d\rho}\\ \mathrm{Q}&=\qquad\ \ -\frac{d\psi}{\rho d\phi}\\ \mathrm{R}&=-\omega\rho\alpha-\frac{d\psi}{dz}\,.\end{aligned}\right\}\quad \ldots\ldots\ldots\ldots \quad (5)$$

The external magnetic force may be represented by

$$\Omega_0=\frac{d\mathrm{P}_0}{dz},\ \mathrm{F}_0=-\frac{d\mathrm{P}_0}{\rho d\phi},\ \mathrm{G}_0=\frac{d\mathrm{P}_0}{d\rho},\ \mathrm{H}_0=0,$$

and all the conditions of the problem are satisfied by taking for the vector potential of the currents in the solid

$$\mathrm{F}=-\frac{d\mathrm{P}}{\rho d\phi},\ \mathrm{G}=\frac{d\mathrm{P}}{d\rho},\ \mathrm{H}=0.$$

The total magnetic force is the sum of the parts due to the currents in the sheet and the external force: it is therefore given by

$$\alpha=-\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{d\rho dz},\ \beta=-\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{\rho d\phi dz},\ \gamma=\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d(\mathrm{P}+\mathrm{P}_0)}{d\rho}\right)+\frac{1}{\rho^2}\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{d\phi^2}.$$

The currents may be denoted by

$$u=-\frac{d\Phi}{\rho d\phi},\ v=\frac{d\Phi}{d\rho},\ w=0,$$

$$4\pi\Phi=-\nabla^2\mathrm{P} \quad \ldots\ldots\ldots\ldots \quad (6_1)$$

If we now substitute for P, Q, R their equivalents σu, σv, 0, the second of equations (5) becomes

$$\sigma\frac{d\Phi}{d\rho}=-\frac{d\psi}{\rho d\phi} \quad \ldots\ldots\ldots\ldots \quad (6_2)$$

which is satisfied by

$$\Psi=\rho\alpha\frac{d\chi}{d\rho},\quad \Phi=-\frac{d\chi}{d\phi}=-\frac{1}{4\pi}\nabla^2\mathrm{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6_3)$$

and from the last equation

$$\omega\rho\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{d\rho dz}-\sigma\rho\frac{d^2\chi}{d\rho dz}=0,$$

whence

$$\sigma\chi=\omega(\mathrm{P}+\mathrm{P}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By equation (6_1) this may be written

$$\frac{\sigma}{4\pi}\nabla^2\mathrm{P}=\omega\frac{d}{d\phi}(\mathrm{P}+\mathrm{P}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is the characteristic equation of steady currents in a rotating conductor: we may show that the first of equations (5) is also satisfied, for this requires that

$$-\frac{\sigma}{\rho}\frac{d\Phi}{d\phi}=\omega\frac{d}{d\rho}\cdot\rho\frac{d(\mathrm{P}+\mathrm{P}_0)}{d\rho}+\frac{\omega}{\rho}\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{d\phi^2}-\omega\frac{d}{d\rho}\cdot\rho\frac{d}{d\rho}(\mathrm{P}+\mathrm{P}_0),$$

or

$$-\sigma\Phi=\omega\frac{d}{d\phi}(\mathrm{P}+\mathrm{P}_0),$$

which corresponds with the former equations.

All the conditions of the problem will be satisfied by determining P subject to the following conditions:—

(1.) It must satisfy equation (8) within the solid, and vanish when $z=-\infty$.

(2.) ,, ,, $\nabla^2\mathrm{P}=0$ outside the solid, and vanish when $z=+\infty$.

(3.) P and $\frac{d\mathrm{P}}{dz}$ must be the same outside and inside the solid when $z=0$.

Let us now suppose that P_0 can be expanded within the solid in a series of the form $\mathrm{P}_0=\Sigma(\mathrm{A}e^{\iota m\phi}+\mathrm{A}'e^{-\iota m\phi})\mathrm{J}_m(\kappa\rho)e^{\kappa z}$: each of these terms verifying the equation $\nabla^2\mathrm{P}_0=0$ as they ought to do, and vanishing when $z=-\infty$. We shall show presently how this may be done.

The corresponding part of P due to $\mathrm{A}e^{\iota m\phi}$ is

$$\mathrm{P}=-\mathrm{A}e^{\iota m\phi}\mathrm{J}_m(\kappa\rho)e^{\kappa z}+\Pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$\frac{\sigma}{4\pi}\nabla^2\Pi=+\omega\frac{d\Pi}{d\phi}.$$

Putting therefore

$$\Pi=\mathrm{B}e^{m\iota\phi}\mathrm{J}_m(\kappa\rho)e^{\mu z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

we have

$$-\kappa^2+\mu^2=i.\frac{4\pi m\omega}{\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\therefore \mu=\pm\sqrt{\kappa^2-i\frac{4\pi m\omega}{\sigma}}=\pm(p-iq) \text{ suppose.}$$

We must take only the positive sign since $\Pi=0$ when $z=-\infty$.

Thus

$$P=e^{im\phi}J_m(\kappa\rho)(-Ae^{\kappa z}+Be^{\mu z}) \text{ within the solid;}$$

outside

$$P=e^{im\phi}J_m(\kappa\rho).Ce^{-\kappa z},$$

and the conditions of continuity give us

$$-A+B=C,\ -\kappa A+\mu B=-\kappa C,$$

whence

$$B=\frac{2\kappa}{\kappa+\mu}.A,\ C=\frac{\kappa-\mu}{\kappa+\mu}.A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Let us confine our attention now to the value of C which expresses the action of the sheet on external points

$$C=A.\frac{\kappa-p+qi}{\kappa+p-qi}=A(p'+q'i).$$

similarly, corresponding to the term $A'e^{-im\phi}$, we find

$$C'=A'(p'-q'.i).$$

Let

$$Ae^{im\phi}+A'e^{-im\phi}=\mathfrak{A}\cos m\phi+\mathfrak{B}\sin m\phi$$
$$Ce^{im\phi}+C'e^{-im\phi}=\mathfrak{C}\cos m\phi+\mathfrak{D}\sin m\phi$$

$$\left.\begin{aligned}\mathfrak{C}&=C+C'=\mathfrak{A}p'+\mathfrak{B}q'\\ \mathfrak{D}&=(C-C')i=\mathfrak{B}p'-\mathfrak{A}q'\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Let

$$p'+q'i=M'.e^{i\vartheta},$$

then corresponding to a term in

$$P_0=(\mathfrak{A}\cos m\phi+\mathfrak{B}\sin m\phi)e^{\kappa z}J_m(\kappa\rho),$$

there exists in P (outside) the term

$$[\mathfrak{A}M'\cos(m\phi+\vartheta')+\mathfrak{B}M'\sin(m\phi+\vartheta')]e^{-\kappa z}J_m(\kappa\rho) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the term is therefore altered in intensity in the ratio M′ : 1 and the azimuth increased by the angle ϑ'. If we put

$$\mu^2 = \mathrm{M}^2 e^{-2i\vartheta},$$

we can easily prove that

$$\mathrm{M}' = \frac{q}{p+q} = \frac{\sin\vartheta}{\sin\vartheta + \cos\vartheta},\ \vartheta' = \frac{\pi}{2} + \tan^{-1}\left(\frac{\mathrm{M}\sin\vartheta}{\kappa}\right)$$

where also

$$\tan 2\vartheta = \frac{4\pi m\omega}{\sigma},\ \mathrm{M}^4 = \kappa^4 + \left(\frac{4\pi m\omega}{\sigma}\right)^2.$$

The action of the solid will therefore be the reverse of the original magnetisation and M′ is a proper fraction.

The coefficients $\mathfrak{A}$, $\mathfrak{B}$ are found by considering the value of P_0 when $z=0$; call it Z_0. Then as we have found

$$Z_0 = \frac{1}{\pi}\Sigma\int_0^{2\pi}\cos m(\phi-\phi')d\phi'\int_0^{\infty}\kappa d\kappa \mathrm{J}_m(\kappa\rho)\int_0^{\infty}Z'_0\mathrm{J}_m(\kappa\rho')\rho' d\rho' . \quad . \quad . \quad . \quad . \ (15)$$

Plate of finite or infinitesimal thickness.

§ 17. For the case of a plate bounded by two parallel planes at distance b apart, we may satisfy all the conditions by taking the same general forms for the vector potentials and for the currents in the sheet: and the characteristic equation for the determination of P will remain the same as before. If we suppose the inducing magnetism distributed on the positive side of the plate, we may express P_0 within the plate as a series of terms of the form

$$\mathrm{A}e^{im\phi}\mathrm{J}_m(\kappa\rho)e^{\kappa z},$$

the origin being in the axis of revolution, and on the positive surface of the plate. The forms for the vector potential due to the currents are all given by taking for P a series of terms of the type

i. In the substance of plate $-\mathrm{A}e^{im\phi}\mathrm{J}_m(\kappa\rho)e^{\kappa z} + e^{im\phi}\mathrm{J}_m(\kappa\rho)(\mathrm{B}_1 e^{\mu z} + \mathrm{B}_2 e^{-\mu z})$
ii. Outside the plate, z positive, $e^{im\phi}\mathrm{J}_m(\kappa\rho)\mathrm{C}e^{-\kappa z}$
iii. Outside, z negative, $e^{im\phi}\mathrm{J}_m(\kappa\rho).\mathrm{D}e^{+\kappa z}$

where

$$\mu = \sqrt{\kappa^2 - i\frac{4\pi\omega m}{\sigma}}.$$

On the two faces of the plate P and $\frac{dP}{dz}$ are to change continuously on crossing the faces.

The final result of the calculation gives

$$C=A.\frac{(\kappa^2-\mu^2)(e^{\mu b}-e^{-\mu b})}{(\kappa+\mu)^2e^{\mu b}-(\kappa-\mu)^2e^{-\mu b}}\quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

when b is indefinitely small, we treat $R=\frac{\sigma}{b}$ as finite, and therefore while κ is finite μ is infinitely great, but μ^2b is finite and equal to $-\frac{4\pi m\omega}{R}.i$. To find C, we have then

$$\frac{C}{C+A}=\frac{\kappa^2-\mu^2}{2\kappa}.\frac{e^{\mu b}-e^{-\mu b}}{\kappa(e^{\mu b}-e^{-\mu b})+\mu(e^{\mu b}+e^{-\mu b})}$$

$$=\frac{(\kappa^2-\mu^2)b}{2\kappa}.\frac{1+\frac{\mu^2b^2}{6}+\ldots}{1+\kappa b+\frac{\mu^2b^2}{2}+\frac{\kappa b.\mu^2b^2}{6}+\ldots}$$

$$=\frac{2\pi\omega m}{\kappa R}.i\quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

If Q be the value (due to a positive image of P_0) on the positive side of the plate, and P the value of P due to the currents,

$$Q=\Sigma Ae^{im\phi}J_m(\kappa\rho)e^{+\kappa z},\quad P=\Sigma Ce^{im\phi}J_m(\kappa\rho)e^{-\kappa z}$$

then, putting

$$\varpi=\frac{2\pi\omega}{R}$$

$$\frac{dP}{dz}-\varpi\frac{dP}{d\phi}=\varpi\frac{dQ}{d\phi}\quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18)$$

This corresponds with MAXWELL's result; see also equation (34).

Spherical shell and sphere.

§ 18. We shall now treat the case of a spherical shell, whose outer and inner radii we shall take to be b, a.

The expressions for the electromotive force are (taking dx, dy, dz to correspond respectively with dr, $rd\theta$, $r\sin\theta d\phi$).

$$\left.\begin{aligned} P &= -\omega r\sin\theta.\beta-\frac{d\psi}{dr} \\ Q &= +\omega r\sin\theta.\alpha-\frac{d\psi}{rd\theta} \\ R &= \qquad\qquad -\frac{d\psi}{r\sin\theta d\phi} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (19)$$

Since, with the present notation, we have $U=0$, $V=0$, $W=\omega r\sin\theta$, we shall choose the same forms for the vector potentials and current-functions as in § 8, viz. :—

$$\Omega_0=\frac{d}{dr}(P_0r),\ F_0=0,\ G_0=-\frac{dP_0}{\sin\theta d\phi},\ H_0=\frac{dP_0}{d\theta},$$

$$F=0,\ G=-\frac{dP}{\sin\theta d\phi},\ H=\frac{dP}{d\theta}.$$

The magnetic force α, β, γ is due to both potentials, and thus (see § 3 B, c),

$$\alpha=\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d\overline{P+P_0}}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2\overline{P+P_0}}{d\phi^2}\right],\ \beta=-\frac{1}{r}\frac{d^2\,\overline{P+P_0}\,r}{drd\theta},\ \gamma=-\frac{1}{r\sin\theta}\frac{d^2.\overline{P+P_0}\,r}{drd\phi}.$$

The currents will be denoted by

$$u=0,\ v=-\frac{d\Phi}{\sin\theta d\phi},\ w=\frac{d\Phi}{d\theta},$$

where

$$4\pi\Phi=-\nabla^2P \quad \ldots\ldots\ldots \quad (20)$$

observing that

$$\nabla^2P_0=0.$$

Substituting for P, Q, R in terms of u, v, w by OHMS' Law, the last of equations (19) becomes

$$\sigma\frac{d\Phi}{d\theta}=-\frac{d\psi}{r\sin\theta d\phi}. \quad \ldots\ldots\ldots \quad (21_1)$$

This is satisfied by taking

$$\Phi=-\frac{d\chi}{d\phi},\ \psi=\sigma.r\sin\theta\frac{d\chi}{d\theta} \quad \ldots\ldots\ldots \quad (21_2)$$

and the first of equations (19) becomes

$$\omega\sin\theta\frac{d^2\overline{P+P_0}\,r}{drd\theta}-\sigma\sin\theta\frac{d^2.\chi r}{drd\theta}=0,$$

or

$$\sigma\chi=\omega.(P+P_0) \quad \ldots\ldots\ldots \quad (21_3)$$

Equation (20) now becomes

$$\frac{\sigma\nabla^2\mathrm{P}}{4\pi}=\omega\frac{d(\mathrm{P}+\mathrm{P}_0)}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

the same characteristic equation as in § 16.

We may now show that these results verify the second of (19) as the reader may easily convince himself by substitution.

(*a.*) When the inducing magnetism or currents are outside the shell altogether, P_0 will consist of a series of terms of the form

$$\mathrm{A}e^{im\phi}\mathrm{P}_m{}^n.\left(\frac{r}{\mathrm{b}}\right)^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and for the value of P we shall have

$$\left.\begin{array}{ll} \text{i. in substance of shell } (\mathrm{b}>r>\mathrm{a}), & \mathrm{P}=\left[-\mathrm{A}\left(\frac{r}{\mathrm{b}}\right)^n+\mathrm{B}_1\mathrm{S}_n(\lambda r)+\mathrm{B}_2\mathrm{T}_n(\lambda r)\right]e^{im\phi}\mathrm{P}_m{}^n \\ \text{ii. outside shell, } r>\mathrm{b} & \mathrm{P}=\mathrm{C}.\left(\frac{\mathrm{b}}{r}\right)^{n+1}e^{im\phi}\mathrm{P}_m{}^n \\ \text{iii. within shell, } r<\mathrm{a} & \mathrm{P}=\mathrm{D}.\left(\frac{r}{\mathrm{b}}\right)^n e^{im\phi}\mathrm{P}_m{}^n \end{array}\right\} \quad . \quad (24)$$

and where

$$\lambda^2=-i.\frac{4\pi m\omega}{\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

we shall put as before

$$x=\lambda r,\ \alpha=\lambda\mathrm{a},\ \beta=\lambda\mathrm{b},\ \mathrm{S}_n(\alpha)=\mathrm{S},\ \mathrm{S}_n(\beta)=\mathrm{S}'_n\ .\ .$$

When $r=\mathrm{a}$,

$$-\mathrm{A}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)^n+\mathrm{B}_1\mathrm{S}_n+\mathrm{B}_2\mathrm{T}_n=\mathrm{D}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)^n,$$

$$-\frac{n}{\lambda}.\mathrm{A}\frac{\mathrm{a}^{n-1}}{\mathrm{b}^n}+\mathrm{B}_1\frac{d\mathrm{S}_n}{d\alpha}+\mathrm{B}_2\frac{d\mathrm{T}_n}{d\alpha}=\frac{n\mathrm{D}}{\lambda}.\frac{\mathrm{a}^{n-1}}{\mathrm{b}^n},$$

and when $r=\mathrm{b}$,

$$-\mathrm{A}+\mathrm{B}_1\mathrm{S}'_n+\mathrm{B}_2\mathrm{T}'_n=\mathrm{C},$$

$$-\frac{n}{\lambda\mathrm{b}}\mathrm{A}+\mathrm{B}_1\frac{d\mathrm{S}'n}{d\beta}+\mathrm{B}_2\frac{d\mathrm{T}'_n}{d\beta}=-(n+1)\frac{\mathrm{C}}{\lambda\mathrm{b}}.$$

By the help of the formulæ of reduction in § 10, we can put the result of the calculation of the new coefficients B_1, B_2, C, D into somewhat elegant forms.

Eliminating C and D we find

$$\mathrm{B}_1\mathrm{S}_{n+1}+\mathrm{B}_2\mathrm{T}_{n+1}=0$$

$$\mathrm{B}_1\mathrm{S}'_{n-1}+\mathrm{B}_2\mathrm{T}'_{n-1}=-\frac{2n+1}{\beta},$$

whence we find for C, which is what we chiefly want,

$$\mathrm{C}=\mathrm{A}\left\{-1-\frac{2n+1}{\beta}\cdot\frac{\mathrm{S}_{n+1}\mathrm{T}'_n-\mathrm{T}_{n+1}\mathrm{S}'_n}{\mathrm{S}_{n+1}\mathrm{T}'_{n-1}-\mathrm{T}_{n+1}\mathrm{S}'_{n-1}}\right\}=\mathrm{A}.\frac{\mathrm{S}_{n+1}\mathrm{T}'_{n+1}-\mathrm{T}_{n+1}\mathrm{S}'_{n+1}}{\mathrm{S}_{n+1}\mathrm{T}'_{n-1}-\mathrm{T}_{n+1}\mathrm{S}'_{n-1}}. \quad . \quad . \quad (26)$$

(*b*.) When the inducing magnetism lies partly within the shell (not in its substance) we may take for P_0 a series of the form

$$\mathrm{A}'e^{im\phi}\mathrm{P}_m{}^n\left(\frac{\mathrm{b}}{r}\right)^{n+1};$$

the other expressions for P will retain their forms except that we must take the first term $-\mathrm{A}'\left(\frac{\mathrm{b}}{r}\right)^{n+1}$ instead of $-\mathrm{A}\left(\frac{r}{\mathrm{b}}\right)^n$. The calculation of C is similar to that given above; and, when the algebraic work is performed, we obtain finally

$$\mathrm{C}=\mathrm{A}'\left\{-1-\frac{2n+1}{\lambda\mathrm{a}}\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^{n+1}\frac{\mathrm{S}'_n\mathrm{T}'_{n-1}-\mathrm{T}'_n\mathrm{S}'_{n-1}}{\mathrm{S}_{n+1}\mathrm{T}'_{n-1}-\mathrm{T}_{n+1}\mathrm{S}'_{n-1}}\right\} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

There are two particular cases of these formulæ which possess a special interest.

(I.) In the case of a *solid sphere* $\mathrm{a}=0$, $\mathrm{S}_{n+1}(\lambda\mathrm{a})=0$.

The inducing magnetism can only belong to type (*a*) and may be represented by $\mathrm{P}_0=\Sigma\mathrm{A}e^{im\phi}\mathrm{P}_m{}^n\left(\frac{r}{\mathrm{b}}\right)^n$: the external effect of the currents is then given, according to (26), by

$$\mathrm{P}=\Sigma\mathrm{A}.\frac{\mathrm{S}_{n+1}(\lambda\mathrm{b})}{\mathrm{S}_{n-1}(\lambda\mathrm{b})}e^{im\phi}\mathrm{P}_m{}^n.\left(\frac{\mathrm{b}}{r}\right)^{n+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and it is easy to deduce from this expression the value of P when P_0 is given by terms which are real in ϕ.

(II.) In the case of an *infinitely thin shell*, we must put $\mathrm{b}=\mathrm{a}+c$.

This case is interesting as it constitutes an extension of MAXWELL's result for a plane plate. A glance at the preceding formulæ will show that we have here to expand $\mathrm{S}_\nu(\lambda\mathrm{a}+\lambda c)$, $\mathrm{T}_\nu(\lambda\mathrm{a}+\lambda c)$, where $\nu=n-1$, n or $n+1$, in ascending powers of c. Some caution is requisite, however, in doing so on account of the value of λ: for

$\lambda^2=-\frac{4\pi mw}{\sigma}.i$, and if we put $\sigma=\mathrm{R}c$, and treat R as finite, we must suppose λ^2c finite, and therefore λc is a small quantity of the order $c^{\frac{1}{2}}$, while $\frac{1}{\lambda}$ is also of the order $c^{\frac{1}{2}}$.

The expression

$$\mathrm{S}_{n+1}\mathrm{T}'_{\nu}-\mathrm{T}_{n+1}\mathrm{S}'_{\nu}$$

$$=\mathrm{S}_{n+1}\mathrm{T}_{\nu}-\mathrm{T}_{n+1}\mathrm{S}_{\nu}+\lambda c\left(\mathrm{S}_{n+1}\frac{\mathrm{T}_{\nu}}{d\alpha}-\mathrm{T}_{n+1}\frac{d\mathrm{S}_{\nu}}{d\alpha}\right)+\tfrac{1}{2}\lambda^2c^2\left(\mathrm{S}_{n+1}\frac{d^2\mathrm{T}_{\nu}}{d\alpha^2}-\mathrm{T}_{n+1}\frac{d^2\mathrm{S}_{\nu}}{d\alpha^2}\right)+\ldots$$

If we write

$$\mathrm{S}_{n+1}\frac{d^p\mathrm{T}_{\nu}}{d\alpha^p}-\mathrm{T}_{n+1}\frac{d^p\mathrm{S}_{\nu}}{d\alpha^p}=u_p,$$

we can readily find a formula of reduction in u_p.

S_{ν} and T_{ν} both satisfy the equation

$$x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}\{x^2-\nu(\nu+1)\}y=0.$$

By LEIBNITZ' theorem

$$x^2y^{(p+2)}+2(p+1)xy^{(p+1)}+(x^2+(p-\nu)(p+\nu+1))y^{(p)}+2py^{(p-1)}+p(p-1)y^{(p-2)}=0,$$

wherein

$$y^{(p)}=\frac{d^py}{dx^p};$$

whence

$$u_{p+2}+\frac{2(p+1)}{x}u_{p+1}+\left(1+\frac{(p-\nu)(p+\nu+1)}{x^2}\right)u_p+\frac{2p}{x^2}u_{p-1}+\frac{p(p-1)}{x^2}u_{p-2}=0,$$

in which we have to write α for x in the problem before us. We conclude that u_{p+2} will usually be of the same degree in $\frac{1}{\alpha}$ as u_p, and when u_0 and u_1 are given we can readily find the remaining values of u_p. But as each of them is multiplied by ever increasing powers of λc, it will be only the first one or two terms which will give finite terms when we treat $\frac{c}{\mathrm{a}}$ as infinitely small. Let us now take up cases (*a*) and (*b*) of last article.

(*a*.) From formula (26) and remembering (B_0), we obtain

$$\frac{\mathrm{C}}{\mathrm{C}+\mathrm{A}}=-\frac{\beta}{2n+1}\cdot\frac{\mathrm{S}_{n+1}\mathrm{T}'_{n+1}-\mathrm{T}_{n+1}\mathrm{S}'_{n+1}}{\mathrm{S}_{n+1}\mathrm{T}'_n-\mathrm{T}_{n+1}\mathrm{S}'_n};$$

now

$$S_{n+1}T'_{n+1}-T_{n+1}S'_{n+1}=\lambda c.\left(S_{n+1}\frac{dT_{n+1}}{d\alpha}-T_{n+1}\frac{dS_{n+1}}{d\alpha}\right)+\ldots$$

$$=\lambda c.-\frac{1}{\lambda^2 a^2}+\text{terms negligible.}$$

$$S_{n+1}T'_{n}-T_{n+1}S'_{n}=S_{n+1}T_{n}-T_{n+1}S_{n}+\lambda c\left(S_{n+1}\frac{dT_{n}}{d\alpha}-T_{n+1}\frac{dS_{n}}{d\alpha}\right)+\ldots$$

$$=\frac{1}{\lambda^2 a^2}+\ldots;$$

and it may be easily shown, by working out the different coefficients of $c, c^2, \ldots$, that the terms neglected are really negligible.

Finally, we obtain

$$\frac{C}{C+A}=\frac{\lambda^2 ac}{2n+1}, \text{ putting a for } b \text{ in } \beta.$$

$$=-\frac{4\pi m\omega a}{(2n+1)R}.\iota \quad \ldots\ldots\ldots\ldots\ldots \quad (29)$$

(*b*.) Here

$$\frac{C+A'}{A'}=-\frac{2n+1}{\lambda a}\left(\frac{b}{a}\right)^{n+1}.\frac{S'_nT'_{n-1}-T'_nS'_{n-1}}{S_{n+1}T'_{n-1}-T_{n+1}S'_{n-1}}.$$

The numerator is, as we know, $\frac{1}{\lambda^2 b^2}$; and the denominator

$$=S_{n+1}T_{n-1}-T_{n+1}S_{n-1}+\lambda c\left(S_{n+1}\frac{dT_{n-1}}{d\alpha}-T_{n+1}\frac{dS_{n-1}}{d\alpha}\right)+\ldots$$

the former of which set of terms $=-\frac{2n+1}{\alpha^3}$.

To find the latter we observe that $\frac{dS_{n-1}}{d\alpha}=\overline{n-1}\,\frac{S_{n-1}}{\alpha}+S_n$, and similarly for T; hence the term in λc is

$$\lambda c\left[-\frac{(n-1)(2n+1)}{\alpha^4}+\frac{1}{\alpha^2}\right],$$

the first term of which is negligible compared to the second: putting these different results together, and making

$$\left(\frac{b}{a}\right)^{n+1}=1 \text{ and } \beta=\alpha,$$

$$\frac{C+A'}{A'}=-\frac{2n+1}{\alpha}.\frac{\frac{1}{\alpha^2}}{-\frac{2n+1}{\alpha^3}+\frac{\lambda c}{\alpha^2}}=\frac{1}{1-\frac{\lambda^2 ac}{2n+1}};$$

and hence

$$\frac{\mathrm{C}}{\mathrm{C}+\mathrm{A}'}=\frac{\lambda^2 \mathrm{a}c}{2n+1}$$

$$=-\frac{4\pi m\omega \mathrm{a}}{(2n+1)\mathrm{R}}\cdot i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

To interpret these results, let Q_0 be the value of P_0 which would result from the magnetism inside the shell combined with the positive image (at the surface) of the magnetism outside the shell, then

$$\mathrm{Q}_0=\Sigma(\mathrm{A}+\mathrm{A}')\left(\frac{\mathrm{b}}{r}\right)^{n+1}e^{im\phi}\mathrm{P}_m^{\,n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and

$$\mathrm{P}=\Sigma\mathrm{C}\left(\frac{\mathrm{b}}{r}\right)^{n+1}e^{im\phi}\mathrm{P}_m^{\,n}:$$

both the foregoing results merge into the equation

$$r^{\frac{1}{2}}\frac{d}{dr}(\mathrm{P}r^{\frac{1}{2}})=\frac{2\pi\omega\mathrm{a}}{\mathrm{R}}\frac{d}{d\phi}(\mathrm{P}+\mathrm{Q}_0)$$

or putting

$$\mathfrak{P}=\mathrm{P}r^{\frac{1}{2}},\ \mathfrak{Q}=\mathrm{Q}_0r^{\frac{1}{2}},\ \varpi=\frac{2\pi\omega}{\mathrm{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$r\frac{d\mathfrak{P}}{dr}-\varpi\mathrm{a}\frac{d\mathfrak{P}}{d\phi}=\varpi\mathrm{a}\frac{d\mathfrak{Q}}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

When $\mathrm{a}=\infty$, and $r=\mathrm{a}+z$, the above equation becomes

$$\frac{d\mathfrak{P}}{dz}-\varpi\frac{d\mathfrak{P}}{d\phi}=\varpi\frac{d\mathfrak{Q}}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

indicating the spiral trail of images obtained by MAXWELL.

If in the sphere we put $r=e^{\frac{\rho}{\mathrm{a}}}$, we have

$$\frac{d\mathfrak{P}}{d\rho}-\varpi\frac{d\mathfrak{P}}{d\phi}=\varpi\frac{d\mathfrak{Q}}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

which is analogous to MAXWELL's equation, and may be interpreted in a similar manner.

§ 19. We shall briefly verify these results by a direct investigation of the case of an infinitely thin shell.

Putting as before

$$u=0,\ v=-\frac{d\Phi}{\sin\theta d\phi},\ w=\frac{d\Phi}{d\theta}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (36)$$

and denoting the vector potential of the shell by

$$\mathrm{F}=0,\ \mathrm{G}=-\frac{d\mathrm{P}}{\sin\theta d\phi},\ \mathrm{H}=\frac{d\mathrm{P}}{d\theta}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (37)$$

we know (MAXWELL's 'Electricity and Magnetism,' Vol. II., Art. 670) that P is the scalar potential of a shell of matter coinciding with the surface of the sphere where surface density is

$$-\frac{\Phi}{\mathrm{a}}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (38)$$

The equations of the currents on the sheet are

$$\left.\begin{aligned}-\frac{\mathrm{R}}{\sin\theta}\frac{d\Phi}{d\phi}&=\omega r\sin\theta.\alpha-\frac{d\psi}{rd\theta}\\ \mathrm{R}\frac{d\Phi}{d\theta}&=\qquad\qquad\quad-\frac{d\psi}{r\sin\theta d\phi}\end{aligned}\right\}\ .\ .\ .\ .\ .\ .\ .\ (39)$$

wherein, also,

$$\alpha=\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d\overline{\mathrm{P}+\mathrm{P}_0}}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2\overline{\mathrm{P}+\mathrm{P}_0}}{d\phi^2}\right]$$

as in the foregoing articles, and where also r is to be put equal to a.

All these equations are satisfied by

$$\Phi=-\frac{d\chi}{d\phi},\ \psi=\mathrm{Ra}\sin\theta\frac{d\chi}{d\theta},\ \mathrm{R}\chi=\omega(\mathrm{P}+\mathrm{P}_0)\ .\ .\ .\ .\ .\ .\ (40)$$

ψ now referring only to the surface of the sheet.

Outside the sheet $\mathrm{P}_1=\mathrm{C}\left(\frac{\mathrm{a}}{r}\right)^{n+1}e^{im\phi}\mathrm{P}_m^{\ n}$,

Inside ,, $\mathrm{P}_2=\mathrm{C}\left(\frac{r}{\mathrm{a}}\right)^{n}e^{im\phi}\mathrm{P}_m^{\ n}$,

P_1 and P_2 being the values due to the currents in the sheet. At the sheet

$$\left(\frac{dP_1}{dr}-\frac{dP_2}{dr}\right)_{r=a}=-4\pi\Phi$$

$$\therefore\ (2n+1)\frac{C}{a}=4\pi E \quad \text{. . . . .} \quad (41)$$

where

$$\Phi=\Sigma(Ee^{im\phi}P_m^{\ n}).$$

But P_0 being $=\Sigma\mathfrak{A}e^{im\phi}P_m^{\ n}$ at the surface of the sheet, equations (40) give

$$RE=-\omega mi(C+\mathfrak{A}),$$

and therefore

$$C=-\frac{4\pi\omega ma}{2n+1}i.(C+\mathfrak{A}) \quad \text{. . . . .} \quad (42)$$

the same result as obtained in the previous article.

V. *Electrostatic Capacity of Glass, II., and of Liquids.*

By J. HOPKINSON, *M.A., D.Sc., F.R.S.*

I. ELECTROSTATIC CAPACITY OF GLASS.—II.

Received November 3,—Read December 16, 1880.

IN 1877* I had the honour of presenting to the Royal Society the results of some determinations of the specific inductive capacity of glass, the results being obtained with comparatively low electromotive forces and periods of charge and discharge of sensible duration.

In 1878 Mr. GORDON† presented to the Royal Society results of experiments, some of them upon precisely similar glasses, by a quite different method, with much greater electromotive forces and with very short times of charge and discharge. Mr. GORDON's results and my own are compared in the following table :—

	GORDON.		HOPKINSON, 1877.
	Christmas, 1877.	July and Aug., 1879.	
Double extra-dense flint	3·164	3·838	10·1
Extra-dense flint	3·053	3·621	
Light flint	3·013	3·443	6·85
Hard crown	3·108	3·310	

It is quite clear that such enormous differences cannot be due to mere errors of observation; they must arise from a radical defect in one method or the other, or from some property of the material under investigation. I have now repeated my own experiments with greater battery power, and with a new key for effecting the connexions of the condensers, and have obtained substantially the same results as before.

Two hypotheses suggest themselves as to the physical properties of glasses which might, if true, account for the diversity of results :—(i.) In my own earlier experiments a considerable time elapsed, during which some have thought residual charge might flow from the glass condenser and go to swell the capacity determined. Sir W.

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* Phil. Trans., 1878, Part I.

† Phil. Trans., 1879, Part I.

THOMSON had informed me that experiments had proved that the capacity of a good insulating glass is sensibly the same, whether the period of discharge be the ten- or twenty-thousandth of a second, or say one-quarter of a second. This statement has been verified. (ii.) It appeared plausible to suppose that specific inductive capacity of glass was not a constant, but was a function of the electromotive force—in other words, that the ratio $\frac{\text{charge of glass condenser}}{\text{difference of potential}}$ was less when the electromotive force was great than when it was small. This surmise gains some force from Dr. KERR's electro-optical results, which show that electrostatic and optical disturbance of a dielectric are not superposable. It has, however, been submitted to a direct test, with the result that, within the limits tried, specific inductive capacity *is* a constant, and that it is not possible that the discrepancy of experimental results can be thus explained. Finally, I have made a rough model of Mr. GORDON's five-plate balance, and used it to make determinations of specific inductive capacity.

Firstly, a brass plate was tried, and its capacity was found less than unity instead of infinite.

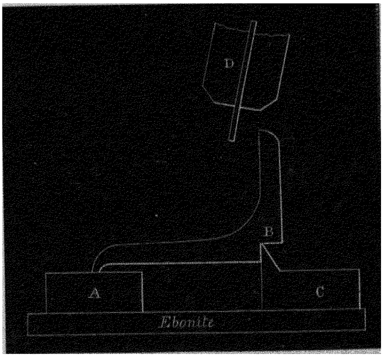
Secondly, by varying the distances of the plates of the balance from each other, different values of the specific inductive capacity of the same glass were obtained. In fact, it has been shown that the five-plate induction balance cannot be freely relied upon to give correct values of specific inductive capacity.

I conclude that the values I published in 1877 are substantially accurate, whether the period of discharge be $\frac{1}{20000}$ or $\frac{1}{2}$ second, whether the electromotive force be one volt per millimetre or 500 volts per millimetre, and that Mr. GORDON's different result is to be explained by a defect in the method he used.

(I.) *To prove that a condenser of well-insulated glass may be almost completely discharged in* $\frac{1}{20000}$ *second.*

For this experiment it is essential that the effect of conduction over the surface of the glass should be insensible. A jar, such as that used in Sir W. THOMSON's electrometer, is unsuitable. The proper form for the condenser is a flask with a thin body and a thick neck, filled with strong sulphuric acid to the neck. Such a flask of light flint glass was prepared, and was instantaneously discharged in the following manner:—The interior of the flask was connected to a metal block, A. Upon this block rests a little L-shaped metal piece, B, which can turn on a knife edge, C. A and C are carried on a block of ebonite, and are therefore insulated. D is a piece of metal connected to earth, and rigidly attached to the extremity of a pendulum. The pendulum is drawn aside and let go; the piece D strikes B and puts the jar to earth, and instantly afterwards breaks the contact with A, and drives away the piece B. In all cases the pendulum was drawn aside 45°, and in all the experiments but one mentioned below it made 93 half-oscillations per minute. The duration of the discharge was determined by the following method, which I arranged for myself, unaware that a similar method had been used by Mr. SABINE.* A condenser of known capacity is connected to A through a known resistance; the condenser receives a known charge whilst connected to the electrometer; the piece B is struck by the pendulum, and the remaining charge is observed. Two experiments were made; in each the condenser was of tinfoil and paraffin, such as are used by Messrs. CLARK, MUIRHEAD, and Co. for telegraph purposes, and had a capacity of 0·29 microfarad. The resistances were respectively 512 ohms and 256 ohms. The results gave respectively duration of discharge 0·0000592 second and 0·0000595 second. We may take it that the duration of discharge was less than 0·00006 second. The condenser was now replaced by the flask. The flask was charged for some seconds from the battery, was insulated and discharged by the pendulum, and the remaining charge read off on the electrometer so soon as the image came to rest. In a first experiment the charge was from four elements (=444 divisions of the scale), and the charge remaining gave deflection 34 divisions. In a second experiment the charge was from eight elements (=888 divisions), and the remaining charge was 61 divisions. Even this small residual charge is largely due to the inductive action of the needle of the electrometer on the quadrant connected to the flask. To prove this, the experiment was varied by beginning with the quadrant separated from the flask, and only connecting these after discharge had been made. With eight battery elements, the remaining charge in the flask was found to be 25 divisions; with 20 elements, 61 divisions. From these experiments we may conclude that, if a flask of light flint glass be charged for some seconds and be discharged for 0·00006 second, the residual charge coming out in the next three or four seconds is certainly less than 3 per cent. of the original charge. It

Fig. 1.



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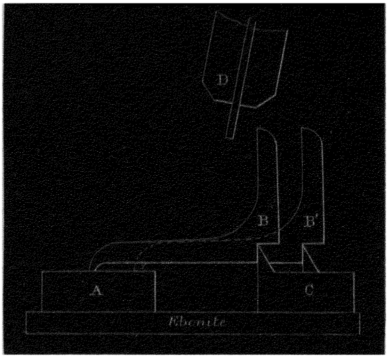
* Philosophical Magazine, May, 1876.

was important to learn if this 3 per cent. was sensibly diminished if the time of discharge was somewhat increased. For this purpose the time of oscillation was increased, and the arrangement of piece B and knife-edge C was duplicated, so that the flask was twice discharged within an interval of about $\frac{1}{80}$ second between. The result was, with charge from eight elements and the flask initially connected to the quadrant, a remaining charge of 61 divisions, exactly the same as when the discharge only lasted $\frac{1}{17000}$ second. I conclude that, with this glass, it matters not whether the discharge of the flask last $\frac{1}{17000}$ second or $\frac{1}{80}$ second; its capacity is the same. This is in precise accord with what Sir W. THOMSON told me before I began the experiments for my former paper.

Fig. 2.



(II.) *Determinations with the guard ring condenser.*[*]

It has been suggested that my former results were liable to uncertainty from the small potentials used and from the comparatively long time of discharge. The main purpose of the present experiments has been to ascertain the force of the objections. As the principle of the method is the same as in the earlier paper, it is only needful to explain the alterations the apparatus have undergone.

The battery.—A chloride of silver battery of 1000 elements was constructed and very carefully insulated, both as regards cell from cell and tray from tray. Each tray contained 50 cells and the set of 20 trays was conveniently enclosed in a wooden case provided with suitable terminals. As my experience of the battery is but short

[*] The cost of the additional instruments used has been defrayed by a Royal Society Grant. The battery and some of the other instruments were made by Messrs. L. CLARK, MUIRHEAD, and Co., the remainder by Mr. GROVES.

I shall not now minutely describe its details, it is sufficient to say that by connecting its middle to earth two condensers can be charged to equal and opposite potentials of 500 elements.

The guard-ring condenser.—This is the instrument of my former experiments, with the switch removed and some slight improvements in mechanical detail. It is by no means perfect in workmanship, and the irregularities of the results now to be given are to be attributed to such imperfections. It was not worth while to make a new instrument, as, for any present interest, determinations of capacities of glasses, correct to 1 per cent., are as valuable as if they were absolutely accurate.

The sliding condensers.—Two sliding condensers were constructed possessing together a very considerable range of capacity. Each has a single scale and is used as before merely as a balance to the guard ring condenser, excepting in one experiment, the subject of the next section.

The switch.—The switch formerly used performed the following operations:—Initially, the quadrant of the electrometer was to earth, the guard ring and the plates of the guard ring condenser were connected to one pole of the battery, the sliding condenser to the other pole. On turning the handle the quadrant and the condensers were insulated; next, the charges of the condensers were mixed, the guard ring being put to earth at the same time; and, finally, the connected condensers were connected to the quadrant of the electrometer; they remained so connected until the handle of the switch was turned back into its first position. This instrument could not be used to

Fig. 3.

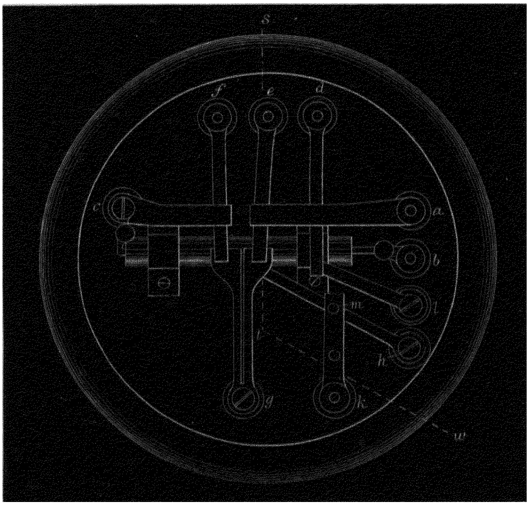
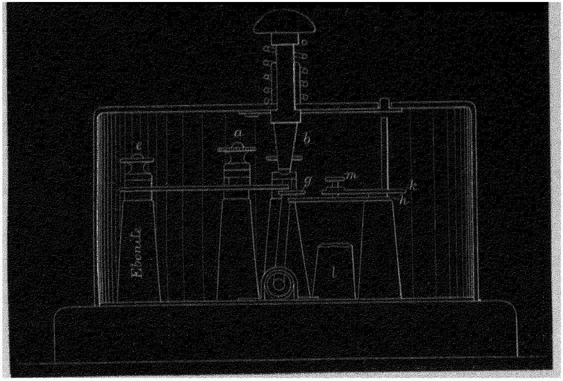


Fig. 4.



Section on line *s*, *t*, *w*.

determine capacities when the residual charge was great, as in the case of plate glass, and was unsatisfactory to anyone who held that flint glass condensers discharged very much more in a time comparable with one second than in a minute fraction of a second. The new switch was arranged to effect the further operation of breaking contact between the condensers and the quadrant immediately after the contact was made. It is also arranged for much higher insulation, the old switch being quite useless for the greater battery power used.

a, *c* are stiff insulated horizontal contact bars connected to the two poles of the battery. *d*, *e*, *f* are insulated springs normally touching *a* and *c* on the under side, *d* is connected by a wire to the guard ring, *e* to the plate of guard ring condenser, *f* to the sliding condenser. *b* is an insulated binding screw connected with *c* for the purpose of more conveniently introducing the battery wire. *l* is a spring connected to earth. *k* is a stiff insulated piece, carrying an adjustable point *m*, normally in contact with the upper side of the insulated spring *h*. From *k* a wire leads to the quadrant of the electrometer. *k* can at any moment be put to earth by a spring key. The insulated spring *g* has its end between *e*, *f*, and *h*, and is normally in contact with neither. The springs *d*, *e*, *f* can be simultaneously bent downwards by an insulated plunger. When this plunger is struck downwards we have the following operations effected in a fraction of a second—

1°. $\begin{cases} d \text{ and } e \text{ are in contact with } a. \\ f \text{ in contact with } c. \end{cases}$

2°. *d*, *e*, and *f* insulated.

3°. $\begin{cases} d \text{ connected to } l. \\ e, f, \text{ and } g \text{ connected together.} \end{cases}$

4°. *e*, *f*, *g*, *h*, *k* connected together.
5°. Connection of *k* and *h* broken.

In other words we have—

1°. Initially $\left\{\begin{array}{l}\text{Guard ring.}\\ \text{Guard ring condenser plate, and}\\ \text{One pole of battery.}\end{array}\right.$
$\left\{\begin{array}{l}\text{Sliding condenser.}\\ \text{Other pole of battery.}\end{array}\right.$
2°. All condensers insulated.
3°. $\left\{\begin{array}{l}\text{Guard ring.}\\ \text{Earth.}\end{array}\right.$
$\left\{\begin{array}{l}\text{Guard ring condenser plate.}\\ \text{Sliding condenser.}\end{array}\right\}$ Mixing of charges.
4°. Mixed charges to electrometer.
5°. Electrometer insulated.

The whole switch, binding screws and all, is covered with a brass cover connected to earth and provided with apertures for the connecting wires. The ebonite legs which carry the pieces *a*, *b*, *c*, *d*, *e* *f*, *g*, *k* are attached to a brass base plate, so that if any leakage occur from *a*, *b*, *c*, *d*, *e*, or *f*, it shall be to earth and not to the electrometer. The connecting wires are insulated with gutta-percha, covered with a metallic tape as an induction shield, this tape being of course connected to earth.

The mode of experiment was substantially as before. A glass plate was introduced in the guard ring condenser, and the sliding condenser adjusted till the capacities were equal; the glass plate was removed and the guard ring condenser, with air as its only dielectric, was adjusted till its capacity was equal to that of the sliding condenser. In every case the battery was reversed and the mean taken.

The following tables give the results obtained:—

All measures are given in terms of turns of the micrometer screw of the guard ring condenser, of which there are 25 to the inch.

Column I. gives the circumstances of the particular experiment.

Column II. the distance between the plates of the condenser with glass in.

Column III. the same distance with air only when the capacity is the same as in II.

Column IV. the thickness of air plate equivalent to glass plate.

Column V. resulting value of K.

DOUBLE extra-dense flint. Density, 4·5. Thickness of plate, 24·27.

I.	II.	III.	IV.	V.
200 elements used, 100 to each condenser, glass in contact with both plates	24·27	2·48	2·48	9·78
1000 elements, contact with both plates	24·27	2·48	2·48	9·78
1000 elements, resting on lower plate, space between glass and upper plate	24·69	2·865	2·445	9·92
Ditto ditto ditto . . .	25·19	3·36	2·44	9·94
Ditto ditto ditto . . .	25·39	3·57	2·45	9·90
Glass separated from lower plate by three small pieces of ebonite also separated from upper plate.	25·19	3 36	2·44	9·94

Mean of last five experiments, K=9·896.
Result formerly published, 10·1.

DENSE flint. Density, 3·66. Thickness of plate, 16·57.

I.	II.	III.	IV.	V.
Glass in contact with both plates, 400 elements	16·57	2 265	2·265	7·31
Glass in contact with both plates, 1000 elements	16·57	2·265	2·265	7 31
Glass resting on lower plate, 1000 elements	17·19	2·85	2·23	7 43
Ditto ditto	17·69	3·36	2·24	7·39

Mean of last three experiments, K=7·376.
Result formerly published, 7·4.

LIGHT flint. Density, 3·2. Thickness of plate, 15·04.

I.	II.	III.	IV.	V.
Glass in contact with both plates, 1000 elements	15·04	2·215	2·215	6·79
Glass resting on lower plate, 1000 elements	15·29	2·505	2·255	6·67
Ditto ditto	15·69	2·865	2·215	6·79
Ditto ditto	16·19	3·42	2·27	6·62

Mean value of K=6·72.
Results formerly published, 6·89 and 6 76=6·83.

Light flint. Thickness, 10·75.

I.	II.	III.	IV.	V.
Contact with both plates, 1000 elements	10·75	1·61	1·61	6·67
Resting on lower plate	11·19	2·035	1·595	6·74
Ditto	11 69	2·555	1·615	6·65

Mean value of K=6·69.

Result formerly published, 6·90.

Mean result for light flint, 6·72.

Mean formerly published, 6·85.

Very light flint. Density, 2·87. Thickness, 12·70.

I.	II.	III.	IV.	V.
Glass in contact with both plates, 400 elements	12·7	1·915	1·915	6·63
Glass in contact with both plates, 1000 elements	12·7	1·915	1 915	6·63
Glass in contact with lower plate only, 1000 elements . . .	12·99	2·215	1·925	6·59
Ditto ditto ditto . . .	13·39	2·61	1·920	6·61

Mean of last three, K=6·61.

Result formerly published, 6·57.

Hard crown. Density, 2·485. Thickness, 11·62.

I.	II.	III.	IV.	V.
Glass in contact with both plates, 1000 elements	11·62	1·675	1·675	6·93
Glass in contact with lower plate only, 1000 elements . . .	11·70	1·74	1·66	7·0
Ditto ditto ditto . . .	11·90	1·945	1·665	6·98
Ditto ditto ditto . . .	12·30	2·255	1·675	6·93

Mean value of K=6·96.

Plate glass. Thickness, 6·52.

I.	II.	III.	IV.	V.
Glass in contact with lower plate only, 400 elements	7·70	1·95	0·77	8·47
Ditto ditto 1000 elements . . .	7·70	1·95	0·77	8·47
Ditto ditto ditto . . .	7·40	1·665	0·785	8·43

Mean value of K=8·45.

REMARK.—On account of the small thickness of the equivalent plate of air, $\frac{1}{30}$ inch, this result is subject to a greater probable error than the others. No inconvenience or uncertainty was experienced from the effect of residual charge. If the screw (*m*, figs. 3, 4, pp. 359, 360) be lowered so that contact with the electrometer is not broken, observation becomes at once impossible.

These results show that my former experiments require no material correction, except in the case of plate glass, for which an accurate experiment was formerly impossible. They also show that electrostatic capacity does not depend on electromotive force up to 200 volts per centimetre for double extra-dense flint, and a somewhat higher electromotive force for the other glasses. It is desirable to show that the same is true for a wider range.

PARAFFIN. Thickness, 20·19.

I.	II.	III.	IV.
Resting on lower	23·82	12·42	8·79
Ditto	22·71	11·32	8·80
Ditto	21·37	9·96	8·78
Contact with both . . .	20·19	8·78	8·78

Mean value of K=2·29.

In this case the guard ring condenser was always charged with 700 elements, the slide with 300 in order that the same sliding condenser might be used.

BOLTZMANN gives 2·32 for paraffin for short times of discharge.

(III.) *To show that* K *is a constant, as is generally assumed.*

Dr. DE LA RUE very kindly allowed me to try a few preliminary experiments last February with his great chloride of silver battery. A flask of extra-dense flint glass was used, insulated with sulphuric acid precisely as in my experiments on residual charge. The comparison was made with a large sliding condenser having a scale graduated in millimetres. Taking one division of the scale (= about 0·0000026 microfarad) as a temporary unit of capacity, I found it impossible to say whether the capacity of the flask was greater or less than 390 divisions, whether the charge in each condenser was 20 elements or 1800 elements. Subsequently a similar experiment was tried with my own battery and a flask of light flint, with the following results, each being the mean of four readings:—

Charge to each condenser in Ag.Cl. elements.	Capacity in millim. divisions of sliding condenser.
10	273·75
100	274·00
200	273·75
300	274·5
400	273·0
500	273·5

The mean of these is 273·75, and the greatest variation from the mean 0·28 per cent.

The conclusion has some considerable importance, for some conceivable molecular theories of specific inductive capacities would lead to the result that capacity would be less when the charge became very great, as is actually the case with the magnetic permeability of iron (*vide* MAXWELL, vol. ii., chapter vi.).

The flasks tried are about 1 millim. thick. I intend to try a very thin glass bulb, testing it to destruction.

[In order to extend the limits of this test, a thin bulb 29 millims. diameter was blown on a piece of thermometer tube and its capacity compared with the sliding condenser with varying charge, as follows:—

100 battery elements to each condenser, capacity of bulb was 297 scale divisions.
300 elements, capacity=297 divisions.
500 elements, capacity=297½ divisions.

The bulb was afterwards broken and the thickness of the fragments measured; they ranged from 0·05 to 0·15 millim., the major portion being about 0·1 millim. We may conclude with confidence that the value of K for the glass tested continues constant up to 5000 volts per millimeter.—Dec. 9, 1880.]

An experiment was subsequently tried to ascertain if specific inductive capacity varied with the temperature of the dielectric. Accurate results could not be obtained, owing to the expansion of the acid, causing it to rise in the neck of the flask. The result of the single experiment tried was, however, that the flask at 14° C. had a capacity equal to 275 divisions of the sliding condenser; at 60° C. it was equal to 280 divisions. Having regard to the increase of capacity due both to the expansion of the glass (which may safely be neglected) and to the expansion of the acid (which is material), we can only conclude that the capacity of glasses certainly does not change rapidly with temperature—that consideration of temperature cannot be expected to reconcile Professor MAXWELL's theory with the results of experiment.

I have repeated the temperature experiment with greater care. The flask was cleansed, filled a little short of the shoulder with acid, and arranged for heating and testing as before. In order to avoid the effect of rising of the level of the acid from expansion, the flask was heated to its highest temperature before any observation.

It was assumed that on cooling the surface of the flask would continue to conduct to the level at which the acid had been.

The following table gives the results of the experiment:—

Temperature Centigrade.	Capacity.	
81°	269½	11th Nov.
48	266	"
27	263½	"
12	262	12th Nov.
39½	266½	"
67½	268½	"
83	271½	"
60	268	"
50½	267	"
13	264	13th Nov.

We may conclude, I think safely, that the specific inductive capacity of light flint does increase slightly, but that the increase from 12° to 83° does not exceed 2½ per cent. The conductivity of the same glass* increases about 100-fold between the same temperature, and the residual charge also increases greatly.

(IV.) *Examination of the method of the five-plate induction balance.*

The theoretical accuracy of this method rests on the assumption that the distance between the plates may be considered small in comparison with their diameter. When this condition is not sufficiently considered, it is easy to see that it is not likely that correct results will in all cases be obtained, for suppose that in lieu of the plate of glass a thin sheet of metal of considerable size is interposed between the fourth and fifth plates of the balance, it ought to be needful to withdraw the fifth plate by an amount equal to the thickness of the sheet. One can apprehend that it will be actually necessary to push it in, but to an extent which it would not be easy to calculate.

Some doubt is also thrown upon the practical accuracy of the method by the fact that Mr. GORDON has arrived at the very unexpected result that the specific inductive capacities of glasses change with the lapse of time.

In order to satisfy myself on the point I had a rough model of a five-plate induction balance made. The instrument is far too rough to give minutely accurate results if the method were good, but I believe it is sufficient to show rapidly that it cannot be used with safety. The insulation was not perfect, and no attempt was made to enclose the instrument or shield the connexions from casual inductive action. The plates are all 4 millims. thick; they are, as in Mr. GORDON'S apparatus, 6 and 4 inches diameter. Each plate is suspended in a vertical plane by two rods and hooks from two of a set of four horizontal rods of varnished glass. The plates can thus be placed parallel to each other at any distance apart that may be desired. The distance between the plates was measured by a pair of common callipers and a millimeter rule to the nearest $\frac{1}{4}$ millimeter. For convenience, let the plates be named A, B, C, D, E, as in the accompanying diagram. In a first experiment B and D were respectively connected to the quadrants of an electrometer of which the jar was charged in the usual way. A and E were connected to one pole of a battery of 20 AgCl elements, C to the other pole through an ordinary electrometer reversing key, E was adjusted till the disturbance of the image was a minimum, when the key was reversed. This method was unsatisfactory, probably because in the act of reversing all the plates A, C, E were momentarily connected to one of the poles, and also because the insulation of the plates B, D was imperfect. The experiments, however, sufficed to prove beyond doubt that the instrument gave diminishing values to the specific inductive capacity of glass, as the distance of the five plates from each other was increased from 12 millims. to 32 millims., also that it gave values less than unity for the specific inductive capacity of brass in the form of a plate 3·5 millims. thick. More satisfactory

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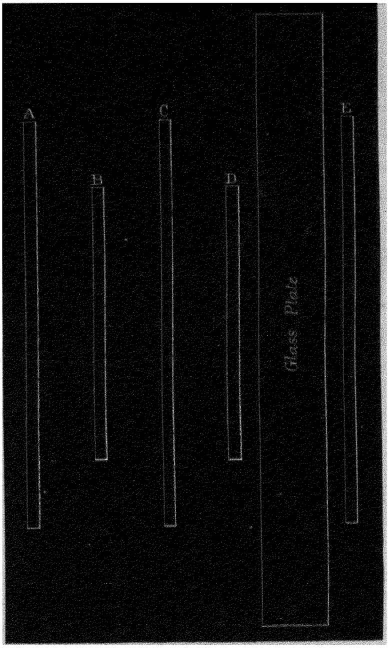
* "Residual Charge of the Leyden Jar," Phil. Trans., 1877.

Fig. 5.



Half full size.

working was attained by approximating, so far as my instruments admitted, to the methods of Mr. GORDON. B and D were, as before, connected to the quadrants, C was connected to the interior of the jar and to one pole of an ordinary induction coil; A and E to the case of the instrument and to the other pole of the induction coil. The plate E was adjusted till the working of the coil caused no deflection of the image on the scale. In each case the plate examined was placed approximately half-way between D and E. The following table gives results of a plate of double extra-dense flint 24·75 millims. thick and 235 millims. diameter, and on a plate of brass 3·5 millims. thick and 242 millims. diameter.

Column I. gives the air-space between the plates AB, BC, or CD.

Column II. the air-space DE (a_1) when no dielectric plate was present.

Column III. the distance DE (a_2) when a dielectric was introduced.

Column IV. the value of the difference $b-(a_2-a_1)$, b being the thickness of the plate, which ought to be constant for each plate.

Column V. the specific inductive capacity $=\frac{b}{b-(a_2-a_1)}$.

DOUBLE extra dense flint, 24·75 m.m. thick.

I.	II.	III.	IV.	V.
5	5¼	27	3	8·25
8	8¼	30¼	2¾	9·0
12	11¾	31¾	4¾	5·21
18	21	37¼	8½	2·91
25	32½	43¾	13½	1·83
32	44½	49½	19¾	1·25

True value of K=9·896.

BRASS plate, 3·5 m.m. thick.

I.	II.	III.	IV.	V.
5	4·5	6·75	1·25	2·8
8	8·0	6·25	5·25	0·66
12	11·25	10·0	4·75	0·73
32	44·5	16·5	31·5	0·11

True value of K= ∞^{ty}.

Inspection of the column IV. shows how impossible it is to attribute the variations of K to any mere error of observation even with the roughest appliances. Column V. demands no comment.

II. ELECTROSTATIC CAPACITY OF LIQUIDS.*

Received January 6,—Read January 27, 1881.

The number of substances suitable for an exact test of Professor MAXWELL'S electromagnetic theory of light is comparatively limited. Amongst solids, besides glass, Iceland spar, fluor spar, and quartz have been examined by ROMICH and NOWAK,† giving results for specific inductive capacity much in excess of the square of the refractive index. On the other hand, the same observers, with BOLTZMANN, obtain for sulphur a value of the capacity in reasonable accord with theory.

On liquids the only satisfactory experiments are those of SILOW‡ on turpentine and petroleum oil, in which the capacity is precisely equal to the square of the refractive index for long waves.

SILOW finally obtains for long waves and capacity—

	μ_∞.	$\sqrt{K}$.
Turpentine	1·461	1·468
Petroleum I.	1·422	1·439
Petroleum II.	1·431	1·428
Benzol	1·482	1·483

A comparison of the whole of the substances which have been examined indicates the generalisation that bodies similar in chemical composition to salts, compounds of an acid, or acids and bases, have capacities much greater than the square of the refractive index, whilst hydrocarbons, such as paraffin and turpentine, cannot be said with certainty to differ from theory one way or the other. It seemed desirable to test this conclusion by experiments on animal and vegetable oils and on other paraffins. It was probable that the compounds of fatty acids and glycerine would have high capacities.

Samples were tested of colza oil, linseed oil, neatsfoot oil, sperm oil, olive oil, castor oil, turpentine, bisulphide of carbon, caoutchoucine, the paraffin actually in use for the electrometer lamp, and three widely different mineral oils kindly given to me by Mr. F. FIELD, F.R.S., to whom I am indebted for the boiling points given below.

The method of experiment was very simple. The sample was first roughly tested for insulation. It was found that it was useless to attempt the samples of colza or linseed oils, of caoutchoucine, or of bisulphide of carbon, but that the rest had sufficient insulation for the tolerably rapid method I was able to use.

The fluid condenser consisted of a double cylinder to contain the fluid, in which an insulated cylinder could hang; three brass rods suspended the latter from an ebonite ring which rested on three legs rising from the outer cylinder of the annular vessel. The position of the insulated cylinder was geometrically determined by three brass stops (*a*, *a*, *a*) which abutted against the legs which carried the ring, six points being thus fixed. A dummy ebonite ring with three brass rods, but without the cylinder, was provided for the purpose of determining the capacity of all parts and connexions not immersed in fluid.

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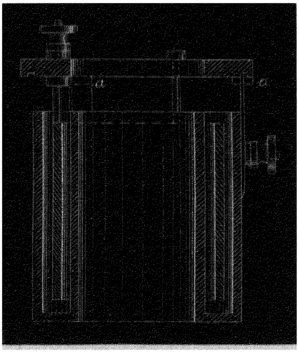
* The abstract of this paper is published in the Proceedings under the title "Dielectric Capacity of Liquids."

† Wiener Sitzb. lxx., part ii., p. 380.

‡ POGG. Ann., 156, 1875, p. 389, and 158, 1876, p. 313.

Fig. 6.



Half full size.

The condenser was balanced against a sliding condenser, first with air and then with fluid.

The key which was used for experiments on plates was used here also, leaving the piece connected to the guard ring idle.

The capacity of the sliding condenser was first tested with the result that to the reading of the slide 82·2 must be added to obtain the capacity in terms of the millimeter divisions of the scale. The capacity of the fluid condenser empty, with its connexions, was 106·5 divisions. The capacity of the dummy and connexions was 7·7, so that the nett capacity of the fluid condenser was 98·8. In all cases 1000 AgCl elements were tried, these being divided between the two condensers.

The following tables give the results obtained:—

Column I. is the number of elements charging the fluid condenser, the complement being used on the sliding condenser.

Column II. the reading of the slide plus 82·2 when a balance was obtained; this is the mean of two readings when the fluid condenser was respectively charged positive and negative.

Column III. is the capacity calculated from the experiment.

Petroleum spirit. Boiling point, 159°.

I.	II.	III.
400	133·2	1·94
500	196·7	1·91
600	294·7	1·91

Mean value of K=1·92.

Petroleum oil (FIELD'S). Boiling point, 310°.

I.	II.	III.
350	114·2	2·07
400	141·2	2·06
500	212·2	2·07

Mean value of K=2·07.

Petroleum oil (*common*).

I.	II.	III.
400	144·2	2·11
500	214·2	2·09
600	321·2	2·09

Mean value of K=2·10.

Ozokerit lubricating oil. Boiling point, 430°.

Two determinations of this oil were made some days apart; at the time of the first determination the oil was slightly turbid. In the interval before determining the refractive index the upper portion became clear, the heavier particles having settled down. The capacity of the clear oil was then determined, and the results are given in the second table. It is possible that if the oil remain quiescent for a longer time a further reduction may be observed.

First experiment.

I.	II.	III.
400	149·2	2·19
500	223·2	2·18
600	334·7	2·18

Mean value of K=2·18.

Second experiment.

I.	II.	III.
400	146·2	2·14
500	217·7	2·12
600	327·7	2·13

Mean value of K=2·13.

Olive oil.

I.	II.	III.
300	137·7	3·17
400	213·7	3·16
500	319·2	3·15

Mean value of K=3·16.

Castor oil.

I.	II.	III.
250	160·2	4·78
300	306·2	4·79
400	319·2	4·76

Mean value of K=4·78.

Sperm oil.

I.	II.	III.
300	132·2	3·04
400	202·7	3·00
500	306·7	3·02

Mean value of K=3·02.

Neatsfoot oil.

I.	II.	III.
300	134·2	3·09
400	206·7	3·06
500	311·2	3·07

Mean value of K=3·07.

Turpentine.

A satisfactory determination for turpentine was not obtained. The turpentine seemed to act on the material of the vessel. After being in the condenser a short time its insulation was much reduced. When the charge had a potential of about 600 elements the condenser discharged itself disruptively through the turpentine. However, with a charge of 100 elements on each condenser a balance was obtained at 228·2, indicating a specific inductive capacity 2·23.

The refractive indices were determined *from the same samples* as the capacities in the usual way by the minimum deviation of a fluid prism. The spectrometer was the same I had previously used for experiments on glass (Proc. Roy. Soc., 1877). The observations were made for the hydrogen lines and the sodium lines, from these the index for long waves was calculated by the formula $a+\frac{b}{\lambda^2}$. The results are given in the following table :—

	μC.	μD.	μF.	μG.	$\mu\,\infty$.	Temperature.
Petroleum spirit	1·3952	1·3974	1·4024	1·4065	1·3865	12·75
Petroleum oil (FIELD'S) . . .	1·4520	1·4547	1·4614	1·4670	1·4406	13·0
Petroleum oil (common) . .	1·4525	1·4551	1·4615	1·4670	1·4416	13·0
Ozokerit lubricating oil . . .	1·4558	1·4585	1·4653	. .	1·4443	13·0
Turpentine	1·4709	1·4738	1·4811	1·4871	1·4586	13·25
Castor oil	1·4785	1·4811	1·4877	1·4931	1·4674	13·5
Sperm oil	1·4724	1·4749	1·4818	. .	1·4611	13·75
Olive oil	1·4710	1·4737	1·4803	. .	1·4598	14·0
Neatsfoot oil	1·4673	1·4696	. .	. .	1·4578	14·0

In the following table is given a synoptic view of the comparison of $\mu\,\infty^2$ and K :—

	$\mu\,\infty$.	$\mu\,\infty^2$.	K.
Petroleum spirit	1·3865	1·922	1·92
Petroleum oil (FIELD'S). . .	1·4406	2·075	2·07
Petroleum oil (common) . .	1·4416	2·078	2·10
Ozokerit lubricating oil . . .	1·4443	2·086	2·13
Turpentine	1·4586	2·128	2·23
Castor oil	1·4674	2·153	4·78
Sperm oil	1·4611	2·135	3·02
Olive oil	1·4598	2·131	3·16
Neatsfoot oil	1·4578	2·125	3·07

A glance shows that vegetable and animal oils do not agree with MAXWELL'S theory, the hydrocarbon oils do. It must, however, never be forgotten that the time of disturbance in the actual optical experiment is many thousands of million times as short as in the fastest electrical experiment even when the condenser is charged or discharged for only the $\frac{1}{20000}$ second.

VI. *On the Forty-eight Coordinates of a Cubic Curve in Space.*

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IN a note published in the Report of the British Association for 1878 (Dublin), and in a fuller paper in the Transactions of the London Mathematical Society, 1879 (vol. x., No. 152), I have given the forms of the eighteen, or the twenty-one (as there explained), coordinates of a conic in space, corresponding, so far as correspondence subsists, with the six coordinates of a straight line in space; and in the same papers I have established the identical relations between these coordinates, whereby the number of independent quantities is reduced to eight, as it should be. In both cases, viz.: the straight line and the conic, the coordinates are to be obtained by eliminating the variables in turn from the two equations representing the line or the conic, and are in fact the coefficients of the equations resulting from the eliminations.

In the present paper I have followed the same procedure for the case of a cubic curve in space. Such a curve may, as is well known, be regarded as the intersection of two quadric surfaces having a generating line in common; and the result of the elimination of any one of the variables from two quadric equations satisfying this condition is of the third degree. The number of coefficients so arising is $4\times10=40$; but I have found that these forty quantities may very conveniently be replaced by forty-eight others, which are henceforward considered as the coordinates of the cubic curve in space. The relation between the forty and the forty-eight coordinates is as follows: on examining the equations resulting from the eliminations of the variables, it turns out that they can be rationally transformed into expressions such as $UP'-U'P=0$, where U and U' are quadrics, and P and P' linear functions of the variables remaining after the eliminations. The forty-eight coordinates then consist of the twenty-four coefficients of the four functions of the form U (say the U coordinates) together with the twenty-four coefficients of the functions of the form U′ (say the U′ coordinates), arising from the four eliminations respectively; viz.: $4\times6+4\times6=48$. And it will be found that the coefficients of the forms P, P′, are already comprised among those of U, U′; so that they do not add to the previous total of forty-eight.

The number of identical relations established in the present paper is 34. But it will be observed that the equations, $UQ'-U'P=0$, are lineo-linear in the U coordinates and in the U′ coordinates; and as we are concerned with the ratios only of the coefficients and not with their absolute values, we are in fact concerned only with the ratios of the U coordinates *inter se*, and of the U′ coordinates *inter se*, and not with their absolute values. Hence the number of independent coordinates will be reduced to $48-34-2=12$, as it should be.

§ 1. *Formation of the equations.*

With a view to the problem in question, it is first required to form the equations of two quadrics having a generating line in common. For the present purpose the following appears the simplest way of effecting this; let

$$\begin{aligned} u&=\alpha x+\beta y+\gamma z+\delta t, & u'&=\alpha' x+\beta' y+\gamma' z+\delta' t, \quad . \quad . \quad . \quad . \quad . \quad (1)\\ v&=\alpha_1 x+\beta_1 y+\gamma_1 z+\delta_1 t, & v'&=\alpha_1' x+\beta_1' y+\gamma_1' z+\delta_1' t,\\ w&=\alpha_2 x+\beta_2 y+\gamma_2 z+\delta_2 t, & w'&=\alpha_2' x+\beta_2' y+\gamma_2' z+\delta_2' t, \end{aligned}$$

From these we may form the three equations

$$\begin{aligned} vw'-v'w&=(a,\ b,\ ..\ )(x,\ y,\ z,\ t)^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)\\ wu'-w'u&=(a_1,\ b_1,\ ..\ )(x,\ y,\ z,\ t)^2=0,\\ uv'-u'v&=(a_2,\ b_2,\ ..\ )(x,\ y,\ z,\ t)^2=0, \end{aligned}$$

of which two only, of course, are independent. Any two of them may be taken as representing the two quadrics in question. Thus, if we take the last two, the equations of the common generating line will be $u=0$, $u'=0$.

The next step is to eliminate the four variables in turn from the two quadrics; for which purpose it will be convenient to express the system (2) in the following form:

$$u:u'=v:v'=w:w'; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or, introducing an indeterminate quantity θ, we may use, instead of equations (3), the following, viz.:

$$u+\theta u'=0, \quad v+\theta v'=0, \quad w+\theta w'=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In order to eliminate one of the variables, say t, let us write,

$$\begin{aligned} u&=u_{\prime}+\delta t, & u'&=u_{\prime}'+\delta' t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)\\ v&=v_{\prime}+\delta_1 t, & v'&=v_{\prime}'+\delta_1' t,\\ w&=w_{\prime}+\delta_2 t, & w'&=w_{\prime}'+\delta_2' t. \end{aligned}$$

The system (4) will then take the form

$$\begin{aligned} u_t+\delta t+\theta u_t'+\theta t\delta' &=0, \\ v_t+\delta_1 t+\theta v_t'+\theta t\delta_1' &=0, \\ w_t+\delta_2 t+\theta w_t'+\theta t\delta_2' &=0; \end{aligned}$$

or, as it may be written,

$$\begin{array}{lllllll} 0=u+\delta t+\theta u'+\theta t\delta' & (u,\ u', & \text{suffix } t; & \delta,\ \delta', & \text{suffix } 0) & \ldots\ldots & (6) \\ 0=v+\delta t+\theta v'+\theta t\delta' & (v,\ v', & \quad ,, \quad t; & \delta,\ \delta', & \quad ,, \quad 1), & & \\ 0=w+\delta t+\theta w'+\theta t\delta' & (w,\ w', & \quad ,, \quad t; & \delta,\ \delta', & \quad ,, \quad 2), & & \end{array}$$

and if we multiply these equations throughout by t, we may write the two systems (6), and t(6) thus:

$$\begin{array}{llllll} u+\theta u'+t\delta+\theta t\delta' \quad \ldots & =0, & (u,\ u', \text{ suffix } t; & \delta,\ \delta', \text{ suffix } 0) & \ldots & (7) \\ v+\theta v'+t\delta+\theta t\delta' \quad \ldots & =0, & (v,\ v', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 1) & & \\ w+\theta w'+t\delta+\theta t\delta' \quad \ldots & =0, & (w,\ w', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 2) & & \\ \ldots\ tu+\theta tu'+t^2\delta+\theta t^2\delta' & =0, & (u,\ u', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 0) & & \\ \ldots\ tv+\theta tv'+t^2\delta+\theta t^2\delta' & =0, & (v,\ v', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 1) & & \\ \ldots\ tw+\theta tw'+t^2\delta+\theta t^2\delta' & =0, & (w,\ w', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 2) & & \end{array}$$

whence we may at once eliminate 1, θ, t, θt, t^2, θt^2, and obtain the following result:

$$\begin{array}{llllll} u,\ u',\ \delta,\ \delta',\ .\ . & =0, & (u,\ u', \text{ suffix } t; & \delta,\ \delta', \text{ suffix } 0) & \ldots & (8) \\ v,\ v',\ \delta,\ \delta',\ .\ . & & (v,\ v', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 1) & & \\ w,\ w',\ \delta,\ \delta',\ .\ . & & (w,\ w', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 2) & & \\ .\ .\ u,\ u',\ \delta,\ \delta' & & (u,\ u', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 0) & & \\ .\ .\ v,\ v',\ \delta,\ \delta' & & (v,\ v', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 1) & & \\ .\ .\ w,\ w',\ \delta,\ \delta' & & (w,\ w', \quad ,, \quad t; & \delta,\ \delta', \quad ,, \quad 2) & & \end{array}$$

The corresponding results, when x, y, z are respectively eliminated, are obvious; viz.: writing down the upper lines only, they may be represented thus:

$$\begin{array}{lllll} \left| u,\ u',\ \alpha,\ \alpha',\ .\ . \right| & =0, & (u,\ u', \text{ suffix } x) & \ldots\ldots & (8') \\ \left| u,\ u',\ \beta,\ \beta',\ .\ . \right| & =0, & (u,\ u', \quad ,, \quad y) & & \\ \left| u,\ u',\ \gamma,\ \gamma',\ .\ . \right| & =0, & (u,\ u', \quad ,, \quad z) & & \end{array}$$

From these equations it is not difficult to write down the coefficients of the powers and products of the variables in each case. Thus in the case of (8), or t eliminated, we shall have, for $[x^3], \dots 3[x^2y], \dots$, *i.e.*, the coefficients of $x^3, \dots 3x^2y, \dots$ respectively,

$$
\begin{array}{rl}
[x^3]= & \alpha,\ \alpha',\ \delta,\ \delta',\ .\ .\ \ (\text{suffix } 0) \\
 & \alpha,\ \alpha',\ \delta,\ \delta',\ .\ .\ \ (\ \ ,,\ \ 1) \\
 & \alpha,\ \alpha',\ \delta,\ \delta',\ .\ .\ \ (\ \ ,,\ \ 2) \\
 & .\ .\ \alpha,\ \alpha',\ \delta,\ \delta',\ (\ \ ,,\ \ 0) \\
 & .\ .\ \alpha,\ \alpha',\ \delta,\ \delta',\ (\ \ ,,\ \ 1) \\
 & .\ .\ \alpha,\ \alpha',\ \delta,\ \delta',\ (\ \ ,,\ \ 2)
\end{array} \qquad (9)
$$

$$
\begin{array}{rllll}
3[x^2y]= & \alpha,\ \alpha',\ \delta,\ \delta',\ .\ . & +\alpha,\ \beta',\ \delta,\ \delta',\ .\ . & +\beta,\ \alpha',\ \delta,\ \delta',\ .\ . & (\text{suffix } 0) \\
 & \alpha,\ \alpha',\ \delta,\ \delta',\ .\ . & \alpha,\ \beta',\ \delta,\ \delta',\ .\ . & \beta,\ \alpha',\ \delta,\ \delta',\ .\ . & (\ \ ,,\ \ 1) \\
 & \alpha,\ \alpha',\ \delta,\ \delta',\ .\ . & \alpha,\ \beta',\ \delta,\ \delta',\ .\ . & \beta,\ \alpha',\ \delta,\ \delta',\ .\ . & (\ \ ,,\ \ 2) \\
 & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & (\ \ ,,\ \ 0) \\
 & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & (\ \ ,,\ \ 1) \\
 & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & .\ .\ \beta,\ \beta',\ \delta,\ \delta' & (\ \ ,,\ \ 2)
\end{array}
$$

But there is another and very useful form which may be given to the equations (8) and (8′). In fact, writing down only the first lines of the determinants, and putting

$$
\begin{array}{lll}
|\,u,\ u',\ \alpha\,| =\mathrm{U}, & |\,u,\ u',\ \alpha'\,| =\mathrm{U}', & (u,\ u',\ \text{suffix } x) \\
|\,u,\ u',\ \beta\,| =\mathrm{V}, & |\,u,\ u',\ \beta'\,| =\mathrm{V}', & (u,\ u',\ \ ,,\ \ y) \\
|\,u,\ u',\ \gamma\,| =\mathrm{W}, & |\,u,\ u',\ \gamma'\,| =\mathrm{W}', & (u,\ u',\ \ ,,\ \ z) \\
|\,u,\ u',\ \delta\,| =\mathrm{T}, & |\,u,\ u',\ \delta'\,| =\mathrm{T}', & (u,\ u',\ \ ,,\ \ t) \\
|\,u,\ \alpha,\ \alpha'\,| =\mathrm{P}, & |\,u',\ \alpha,\ \alpha'\,| =\mathrm{P}', & (u,\ u',\ \ ,,\ \ x) \\
|\,u,\ \beta,\ \beta'\,| =\mathrm{Q}, & |\,u',\ \beta,\ \beta'\,| =\mathrm{Q}', & (u,\ u',\ \ ,,\ \ y) \\
|\,u,\ \gamma,\ \gamma'\,| =\mathrm{R}, & |\,u',\ \gamma,\ \gamma'\,| =\mathrm{R}', & (u,\ u',\ \ ,,\ \ z) \\
|\,u,\ \delta,\ \delta'\,| =\mathrm{S}, & |\,u',\ \delta,\ \delta'\,| =\mathrm{S}', & (u,\ u',\ \ ,,\ \ t)
\end{array} \qquad (10)
$$

We may write the results of eliminating x, y, z, t respectively in the following forms:

$$
\begin{array}{l}
\mathrm{UP}'-\mathrm{U}'\mathrm{P}=0 \\
\mathrm{VQ}'-\mathrm{V}'\mathrm{Q}=0, \\
\mathrm{WR}'-\mathrm{W}'\mathrm{R}=0, \\
\mathrm{TS}'-\mathrm{T}'\mathrm{S}=0;
\end{array} \qquad (11)
$$

and these may be regarded either as the equations of the projections of the curve on the four coordinate planes respectively, or as equations of conical surfaces passing through the curve, and having their axes parallel to the coordinate axes respectively.

It might at first sight appear that, in each of the curves (11), we should in general have nine points determined as follows: viz. (taking the last equation)

by the intersection of S and S′, one point,
,, ,, T ,, S, two points,
,, ,, T′ ,, S′, two points,
,, ,, T ,, T′, four points.

nine in all. This however is not strictly the case, for

$$\begin{array}{rlll}
S', S, T', T = & u,\ u',\ \delta,\ \delta', & (u,\ u',\ \text{suffix } t; & \delta,\ \delta',\ \text{suffix } 0)\quad .\ .\ .\quad (12)\\
& v,\ v',\ \delta,\ \delta', & (v,\ v',\quad \text{,,}\quad t; & \delta,\ \delta',\quad \text{,,}\quad 1)\\
& w,\ w',\ \delta,\ \delta', & (w,\ w',\quad \text{,,}\quad t; & \delta,\ \delta',\quad \text{,,}\quad 2)
\end{array}$$

and since the four determinants which can be formed from the above matrix are equivalent to only two independent determinants, it follows that if any two of the equations $S=0$, $S'=0$, $T=0$, $T'=0$, are satisfied, all will be satisfied. In other words, the four curves S, S′, T, T′, have a common point of intersection.

Instead, however, of taking the coefficients of the equations (8) and (8′), *i.e.*, the quantities of which (9) are specimens, as the forty coordinates of the curve, it will be convenient to take the coefficients of the functions (10) as the coordinates. These will be found to be forty-eight in number, and to be comprised in the following table, as stated in the introductory remarks. For the sake of brevity, only the upper lines of the determinants are written down:

$$\begin{array}{llll}
\beta, \beta', \alpha = B, & (\text{suffix } x), & \beta, \delta', \alpha + \delta, \beta', \alpha = 2M & (\text{suffix } x), \quad .\ .\quad (13)\\
\gamma, \gamma', \alpha = C, & (\ \text{,,}\ \ x), & \gamma, \delta', \alpha + \delta, \gamma', \alpha = 2N & (\ \text{,,}\ \ x),\\
\delta, \delta', \alpha = D, & (\ \text{,,}\ \ x), & \beta, \gamma', \alpha + \gamma, \beta', \alpha = 2F & (\ \text{,,}\ \ x),\\[8pt]
\alpha, \alpha', \beta = A, & (\ \text{,,}\ \ y), & \alpha, \delta', \beta + \delta, \alpha', \beta = 2L & (\ \text{,,}\ \ y),\\
\gamma, \gamma', \beta = C, & (\ \text{,,}\ \ y), & \gamma, \delta', \beta + \delta, \gamma', \beta = 2N & (\ \text{,,}\ \ y),\\
\delta, \delta', \beta = D, & (\ \text{,,}\ \ y), & \gamma, \alpha', \beta + \alpha, \gamma', \beta = 2G & (\ \text{,,}\ \ y),\\[8pt]
\alpha, \alpha', \gamma = A, & (\ \text{,,}\ \ z), & \alpha, \delta', \gamma + \delta, \alpha', \gamma = 2L & (\ \text{,,}\ \ z),\\
\beta, \beta', \gamma = B, & (\ \text{,,}\ \ z), & \beta, \delta', \gamma + \delta, \beta', \gamma = 2M & (\ \text{,,}\ \ z),\\
\delta, \delta', \gamma = D, & (\ \text{,,}\ \ z), & \alpha, \beta', \gamma + \beta, \alpha', \gamma = 2H & (\ \text{,,}\ \ z),\\[8pt]
\alpha, \alpha', \delta = A, & (\ \text{,,}\ \ t), & \beta, \gamma', \delta + \gamma, \beta', \delta = 2F & (\ \text{,,}\ \ t),\\
\beta, \beta', \delta = B, & (\ \text{,,}\ \ t), & \gamma, \alpha', \delta + \alpha, \gamma', \delta = 2G & (\ \text{,,}\ \ t),\\
\gamma, \gamma', \delta = C, & (\ \text{,,}\ \ t), & \alpha, \beta', \delta + \beta, \alpha', \delta = 2H & (\ \text{,,}\ \ t),
\end{array}$$

with a similar set of expressions of which the following are specimens :

$$\begin{array}{lll} \beta, \beta', \alpha' = B'_x, & \beta, \delta', \alpha' + \delta, \beta', \alpha' = 2M'_x & \quad \dots\dots \quad (14) \\ \vdots \quad \vdots & \quad \vdots \qquad\quad \vdots \qquad \vdots & \end{array}$$

Substituting these values in (10), we shall find the following to be the forms of the functions therein contained, viz. :

$$\begin{array}{ll} U = (B, C, D, N, M, F)(y, z, t)^2, & U' = (B', \dots)(y, z, t)^2; \ (B, \dots \text{suffix } x) \quad . \quad (15) \\ V = (C, A, D, L, N, G)(z, x, t)^2, & V' = (C', \dots)(z, x, t)^2; \ (C, \dots \quad ,, \quad y) \\ W = (A, B, D, M, L, H)(x, y, t)^2, & W' = (A', \dots)(x, y, t)^2; \ (A, \dots \quad ,, \quad z) \\ T = (A, B, C, F, G, H)(x, y, z)^2, & T' = (A', \dots)(x, y, z)^2; \ (A, \dots \quad ,, \quad t) \\ P = A_y y + A_z z + A_t t, & P' = A_y' y + \dots) \\ Q = B_x x + B_z z + B_t t, & Q' = B_x' x + \dots) \\ R = C_x x + C_y y + C_t t, & R' = C_x' x + \dots) \\ S = D_x x + D_y y + D_z z, & S' = D_x' x + \dots) \end{array}$$

And finally we may, by means of the equations (11), replace forms having the coefficients (9), by forms having coefficients into which the quantities A, B, ... enter. Thus, writing

$$B_x A_y' - B_x' A_y = (B, A, x, y), \quad C_x A_z' - C_x' A_z = (C, A, x, z), \dots$$

The forms in question are

$$\begin{array}{l} \{(B, A, x, y), \quad \dots\dots\dots\dots \quad (15') \\ \ (C, A, x, z), \\ \ (D, A, x, t), \\ \ (C, A, x, t) + 2(N, A, x, z), \\ \ (D, A, x, z) + 2(N, A, x, t), \\ \ (B, A, x, t) + 2(M, A, x, y), \\ \ (D, A, x, y) + 2(M, A, x, t), \\ \ (B, A, x, z) + 2(F, A, x, y), \\ \ (C, A, x, y) + 2(F, A, x, z), \\ \ (N, A, x, y) + (M, A, x, z) + (F, A, x, t)\} \big)\!\big(y, z, t)^3 \end{array}$$

$$\begin{array}{l}
\{(\mathrm{C},\ \mathrm{B},\ y,\ z), \\
\ (\mathrm{A},\ \mathrm{B},\ y,\ x), \\
\ (\mathrm{D},\ \mathrm{B},\ y,\ t), \\
\ (\mathrm{A},\ \mathrm{B},\ y,\ t)+2(\mathrm{L},\ \mathrm{B},\ y,\ x), \\
\ (\mathrm{D},\ \mathrm{B},\ y,\ x)+2(\mathrm{L},\ \mathrm{B},\ y,\ t), \\
\ (\mathrm{C},\ \mathrm{B},\ y,\ t)+2(\mathrm{M},\ \mathrm{B},\ y,\ z), \\
\ (\mathrm{D},\ \mathrm{B},\ y,\ z)+2(\mathrm{M},\ \mathrm{B},\ y,\ t), \\
\ (\mathrm{C},\ \mathrm{B},\ y,\ x)+2(\mathrm{G},\ \mathrm{B},\ y,\ z), \\
\ (\mathrm{A},\ \mathrm{B},\ y,\ z)+2(\mathrm{G},\ \mathrm{B},\ y,\ x), \\
\ (\mathrm{L},\ \mathrm{B},\ y,\ z)+\ (\mathrm{N},\ \mathrm{B},\ y,\ x)+(\mathrm{G},\ \mathrm{B},\ y,\ t)\}\big)\!\big(z,\ x,\ t)^3
\end{array}$$

$$\begin{array}{l}
\{(\mathrm{A},\ \mathrm{C},\ z,\ x), \\
\ (\mathrm{B},\ \mathrm{C},\ z,\ y), \\
\ (\mathrm{D},\ \mathrm{C},\ z,\ t), \\
\ (\mathrm{B},\ \mathrm{C},\ z,\ t)+2(\mathrm{M},\ \mathrm{C},\ z,\ y), \\
\ (\mathrm{D},\ \mathrm{C},\ z,\ y)+2(\mathrm{M},\ \mathrm{C},\ z,\ t), \\
\ (\mathrm{A},\ \mathrm{C},\ z,\ t)+2(\mathrm{L},\ \mathrm{C},\ z,\ x), \\
\ (\mathrm{D},\ \mathrm{C},\ z,\ x)+2(\mathrm{L},\ \mathrm{C},\ z,\ t), \\
\ (\mathrm{A},\ \mathrm{C},\ z,\ y)+2(\mathrm{H},\ \mathrm{C},\ z,\ x), \\
\ (\mathrm{B},\ \mathrm{C},\ z,\ x)+2(\mathrm{H},\ \mathrm{C},\ z,\ y), \\
\ (\mathrm{M},\mathrm{C},\ z,\ x)+\ (\mathrm{L},\ \mathrm{C},\ z,\ y)+(\mathrm{H},\ \mathrm{C},\ z,\ t)\}\big)\!\big(x,\ y,\ t)^3
\end{array}$$

$$\begin{array}{l}
\{(\mathrm{A},\ \mathrm{D},\ t,\ x), \\
\ (\mathrm{B},\ \mathrm{D},\ t,\ y), \\
\ (\mathrm{C},\ \mathrm{D},\ t,\ z), \\
\ (\mathrm{B},\ \mathrm{D},\ t,\ z)+2(\mathrm{F},\ \mathrm{D},\ t,\ y), \\
\ (\mathrm{C},\ \mathrm{D},\ t,\ y)+2(\mathrm{F},\ \mathrm{D},\ t,\ z), \\
\ (\mathrm{C},\ \mathrm{D},\ t,\ x)+2(\mathrm{G},\ \mathrm{D},\ t,\ z), \\
\ (\mathrm{A},\ \mathrm{D},\ t,\ z)+2(\mathrm{G},\ \mathrm{D},\ t,\ x), \\
\ (\mathrm{A},\ \mathrm{D},\ t,\ y)+2(\mathrm{H},\ \mathrm{D},\ t,\ x), \\
\ (\mathrm{B},\ \mathrm{D},\ t,\ x)+2(\mathrm{H},\ \mathrm{D},\ t,\ y), \\
\ (\mathrm{F},\ \mathrm{D},\ t,\ x)+\ (\mathrm{G},\ \mathrm{D},\ t,\ y)+(\mathrm{H},\ \mathrm{D},\ t,\ z)\}\big)\!\big(x,\ y,\ z)^3
\end{array}$$

§ 2. *The identical relations between the forty-eight coordinates.*

On inspecting the expressions (13) and (14) it will be seen at once that the following relations subsist, viz.:

$$\begin{array}{ll} .\ \ N_y+M_z+F_t=0, & .\ \ N_y'+M_z'+F_t'=0, \\ N_x+\ .\ +L_z+G_t=0, & N_x'+\ .\ +L_z'+G_t'=0, \\ M_x+L_y+\ .\ +H_t=0, & M_x'+L_y'+\ .\ +H_t'=0, \\ F_x+G_y+H_z+\ .\ =0\ ; & F_x'+G_y'+H_z'+\ .\ =0. \end{array} \quad \ldots\ldots \quad (16)$$

This is a set of $4+4=8$ identical relations between the forty-eight coordinates.

Added February 19, 1880.

The next set of identical relations is to be sought among the first minors of the discriminants of U, V, . . . , bordered respectively with the coefficients of P, Q, . . . Thus, selecting T, we have to examine the coefficients of A, B, . . . (say the quantities $\mathfrak{A}$, $\mathfrak{B}$, . . .) in the development of

$$\begin{array}{llll} A, & H, & G, & D_x \\ H, & B, & F, & D_y \\ G, & F, & C, & D_z \\ D_x, & D_y, & D_z & \end{array} \quad (A, B, \ldots, \text{suffix } t).$$

We then have, changing the sign,

$$\begin{aligned} \mathfrak{A} &= BD_z^2-2FD_zD_y+CD_y^2 \\ &= |\beta, \beta', \delta| \times |\delta, \delta', \gamma|^2 \\ &\quad -\{\, |\beta, \gamma', \delta| \times |\gamma, \beta', \delta| \,\}\, |\delta, \delta', \gamma| \times |\delta, \delta', \beta| \\ &\quad +\ |\gamma, \gamma', \delta| \times |\delta, \delta', \beta|^2 \\ &= \{\, |\beta, \beta', \delta| \times |\delta, \delta', \gamma| - |\gamma, \beta', \delta| \times |\delta, \delta', \beta| \,\}\, |\delta, \delta', \gamma| \\ &\quad -\{\, |\beta, \gamma', \delta| \times |\delta, \delta', \gamma| - |\gamma, \gamma', \delta| \times |\delta, \delta', \beta| \,\}\, |\delta, \delta', \beta| \\ &= \begin{array}{llllll} \beta, & \beta', & \delta, & . & . & \gamma \\ \beta_1, & \beta_1', & \delta_1, & . & . & \gamma_1 \\ \beta_2, & \beta_2', & \delta_2, & . & . & \gamma_2 \\ \beta, & . & . & \delta, & \delta', & \gamma \\ \beta_1, & . & . & \delta_1, & \delta_1', & \gamma_1 \\ \beta_2, & . & & \delta_2, & \delta_2', & \gamma_2 \end{array} \times |\delta, \delta', \gamma| \times \begin{array}{llllll} \gamma, & \gamma', & \delta & . & . & \beta \\ \gamma_1, & \gamma_1', & \delta_1, & . & . & \beta_1 \\ \gamma_2, & \gamma_2', & \delta_2, & . & . & \beta_2 \\ \gamma, & . & . & \delta, & \delta', & \beta \\ \gamma_1, & . & . & \delta_1, & \delta_1', & \beta_1 \\ \gamma_2, & . & . & \delta_2, & \delta_2', & \beta_2 \end{array} \end{aligned}$$

and if in these last determinants we subtract rows 4, 5, 6 from rows 1, 2, 3 respectively; and then subtract column 3 from column 4, we shall find that the whole expression

$$= | \beta, \gamma, \delta | \{ | \beta', \delta, \delta' | \times | \gamma, \delta, \delta' | - | \gamma', \delta, \delta' | \times | \beta, \delta, \delta' | \}$$

$$= - | \beta, \gamma, \delta | (\mathrm{D}, \mathrm{D}, y, z),$$

or more simply

$$= | \beta, \gamma, \delta | (\mathrm{D}, y, z).$$

From this result we may at once conclude the following group:

$$\mathfrak{A} = | \beta, \gamma, \delta | (\mathrm{D}, y, z) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (17)$$

$$\mathfrak{B} = | \gamma, \alpha, \delta | (\mathrm{D}, z, x),$$

$$\mathfrak{C} = | \alpha, \beta, \delta | (\mathrm{D}, x, y).$$

Again, proceeding as before we should find

$$-2\mathfrak{F} = 2(\mathrm{A}\mathrm{D}_y\mathrm{D}_z - \mathrm{H}\mathrm{D}_z\mathrm{D}_x - \mathrm{G}\mathrm{D}_x\mathrm{D}_y + \mathrm{F}\mathrm{D}_x^2) \quad (\mathrm{A}, \mathrm{H}, \ldots \text{ suffix } t)$$

$$= \{ | \alpha, \alpha', \delta | + | \alpha, \alpha', \delta | \} | \delta, \delta', \beta | \times | \delta, \delta', \gamma |$$

$$- \{ | \alpha, \beta', \delta | + | \beta, \alpha', \delta | \} | \delta, \delta', \alpha | \times | \delta, \delta', \gamma |$$

$$- \{ | \gamma, \alpha', \delta | + | \alpha, \gamma', \delta | \} | \delta, \delta', \beta | \times | \delta, \delta', \alpha |$$

$$+ \{ | \beta, \gamma', \delta | + | \gamma, \beta', \delta | \} | \delta, \delta', \alpha | \times | \delta, \delta', \alpha |$$

$$= \begin{array}{cccccc} \alpha, & \alpha', & \delta, & \cdot & \cdot & \beta \\ \alpha_1, & \alpha_1', & \delta_1, & \cdot & \cdot & \beta_1 \\ \alpha_2, & \alpha_2', & \delta_2, & \cdot & \cdot & \beta_2 \\ \alpha, & \cdot & \cdot & \delta, & \delta', & \beta \\ \alpha_1, & \cdot & \cdot & \delta_1, & \delta_1', & \beta_1 \\ \alpha_2, & \cdot & \cdot & \delta_2, & \delta_2', & \beta_2 \end{array} \times | \delta, \delta', \gamma | \times \begin{array}{cccccc} \alpha, & \alpha', & \delta, & \cdot & \cdot & \gamma \\ \alpha_1, & \alpha_1', & \delta_1, & \cdot & \cdot & \gamma_1 \\ \alpha_2, & \alpha_2', & \delta_2, & \cdot & \cdot & \gamma_2 \\ \alpha, & \cdot & \cdot & \delta, & \delta', & \gamma \\ \alpha_1, & \cdot & \cdot & \delta_1, & \delta_1', & \gamma_1 \\ \alpha_2, & \cdot & \cdot & \delta_2, & \delta_2', & \gamma_2 \end{array} \times | \delta, \delta', \beta |$$

$$- \left\{ \begin{array}{cccccc} \alpha, & \gamma', & \delta, & \cdot & \cdot & \beta \\ \alpha_1, & \gamma_1', & \delta_1, & \cdot & \cdot & \beta_1 \\ \alpha_2, & \gamma_2', & \delta_2, & \cdot & \cdot & \beta_2 \\ \alpha, & \cdot & \cdot & \delta, & \delta', & \beta \\ \alpha_1, & \cdot & \cdot & \delta_1, & \delta_1', & \beta_1 \\ \alpha_2, & \cdot & \cdot & \delta_2, & \delta_2', & \beta_2 \end{array} + \begin{array}{cccccc} \alpha, & \beta', & \delta, & \cdot & \cdot & \gamma \\ \alpha_1, & \beta_1', & \delta_1, & \cdot & \cdot & \gamma_1 \\ \alpha_2, & \beta_2', & \delta_2', & \cdot & \cdot & \gamma_2 \\ \alpha, & \cdot & \cdot & \delta, & \delta', & \gamma \\ \alpha_1, & \cdot & \cdot & \delta_1, & \delta_1', & \gamma_1 \\ \alpha_2, & \cdot & \cdot & \delta_2, & \delta_2', & \gamma_2 \end{array} \right\} \times | \delta, \delta', \alpha |$$

$$= - | \delta, \delta', \alpha' | \times | \delta, \delta', \gamma | \times | \alpha, \delta, \beta | - | \delta, \delta', \alpha' | \times | \delta, \delta', \beta | \times | \alpha, \delta, \gamma |$$

$$+ | \delta, \delta', \alpha | \times | \delta, \delta', \gamma' | \times | \alpha, \delta, \beta | + | \delta, \delta', \alpha | \times | \delta, \delta', \beta' | \times | \alpha, \delta, \gamma |$$

$$= | \alpha, \delta, \beta | (\mathrm{D}, z, x) + | \gamma, \alpha, \delta | (\mathrm{D}, x, y).$$

From this we may conclude the following group,

$$
\begin{aligned}
2\mathfrak{F} &= |\alpha, \beta, \delta|(\mathrm{D}, z, x) + |\gamma, \alpha, \delta|(\mathrm{D}, x, y) \quad . \quad . \quad . \quad . \quad . \qquad (18)\\
2\mathfrak{G} &= |\alpha, \beta, \delta|(\mathrm{D}, y, z) + |\beta, \gamma, \delta|(\mathrm{D}, x, y),\\
2\mathfrak{H} &= |\gamma, \alpha, \delta|(\mathrm{D}, y, z) + |\beta, \gamma, \delta|(\mathrm{D}, z, x).
\end{aligned}
$$

From (17) and (18) we can now eliminate the three quantities $|\beta, \gamma, \delta|$, $|\gamma, \alpha, \delta|$, $|\alpha, \beta, \delta|$, and obtain the following identical relations,

$$
\begin{aligned}
&\mathfrak{B}(\mathrm{D}, x, y)^2 - 2\mathfrak{F}(\mathrm{D}, x, y)(\mathrm{D}, z, x) + \mathfrak{C}(\mathrm{D}, z, x)^2 = 0 \quad . \quad . \quad . \quad . \qquad (19)\\
&\mathfrak{C}(\mathrm{D}, y, z)^2 - 2\mathfrak{G}(\mathrm{D}, y, z)(\mathrm{D}, x, y) + \mathfrak{A}(\mathrm{D}, x, y)^2 = 0\\
&\mathfrak{A}(\mathrm{D}, z, x)^2 - 2\mathfrak{H}(\mathrm{D}, z, x)(\mathrm{D}, y, z) + \mathfrak{B}(\mathrm{D}, y, z)^2 = 0
\end{aligned}
$$

to these may be added

$$
\begin{aligned}
&-\mathfrak{F}(\mathrm{D}, y, z)^2 - \mathfrak{A}(\mathrm{D}, z, x)(\mathrm{D}, x, y) + \mathfrak{H}(\mathrm{D}, x, y)(\mathrm{D}, y, z) + \mathfrak{G}(\mathrm{D}, y, z)(\mathrm{D}, z, x) = 0 \quad . \quad (20)\\
&-\mathfrak{G}(\mathrm{D}, z, x)^2 + \mathfrak{H}(\mathrm{D}, z, x)(\mathrm{D}, x, y) - \mathfrak{B}(\mathrm{D}, x, y)(\mathrm{D}, y, z) + \mathfrak{F}(\mathrm{D}, y, z)(\mathrm{D}, z, x) = 0\\
&-\mathfrak{H}(\mathrm{D}, x, y)^2 + \mathfrak{G}(\mathrm{D}, z, x)(\mathrm{D}, x, y) + \mathfrak{F}(\mathrm{D}, x, y)(\mathrm{D}, y, z) - \mathfrak{C}(\mathrm{D}, y, z)(\mathrm{D}, z, x) = 0
\end{aligned}
$$

But, as the six equations (19) and (20) are in any case equivalent to only three independent conditions, it is not necessary to go beyond the equations (19).

[POSTSCRIPT.

Added April 23, 1881

In the present case however the equations (19) are themselves not independent, as may be shown in the following way. Write for the moment

$$\mathrm{D}_x = \mathrm{X},\ \mathrm{D}_y = \mathrm{Y},\ \mathrm{D}_z = \mathrm{Z};\ \mathrm{D}_x' = \mathrm{X}',\ \mathrm{D}_y' = \mathrm{Y}',\ \mathrm{D}_z' = \mathrm{Z}';$$

then

$$(\mathrm{D}, y, z) = \mathrm{Y'Z} - \mathrm{YZ'},\ (\mathrm{D}, z, x) = \mathrm{Z'X} - \mathrm{ZX'},\ (\mathrm{D}, x, y) = \mathrm{X'Y} - \mathrm{XY'}.$$

Inserting these values in the first equations of (19) we obtain

$$
\begin{aligned}
&\mathfrak{B}(\mathrm{X'Y} - \mathrm{XY'})^2 - 2\mathfrak{F}(\mathrm{X'Y} - \mathrm{XY'})(\mathrm{Z'X} - \mathrm{ZX'}) + \mathfrak{C}(\mathrm{Z'X} - \mathrm{ZX'})^2 = 0\\
&\quad = (\mathfrak{B}\mathrm{Y}^2 + 2\mathfrak{F}\mathrm{YZ} + \mathfrak{C}\mathrm{Z}^2)\mathrm{X}'^2 + \mathfrak{B}\mathrm{X}^2.\mathrm{Y}'^2 + \mathfrak{C}\mathrm{X}^2.\mathrm{Z}'^2 + 2\mathfrak{F}\mathrm{X}^2.\mathrm{Y'Z'}\\
&\quad - 2\mathfrak{C}\mathrm{ZX.Z'X'} - 2\mathfrak{B}\mathrm{XY.X'Y'} - 2\mathfrak{F}\mathrm{XY.Z'X'} - 2\mathfrak{F}\mathrm{ZX.X'Y'}.
\end{aligned}
$$

But

$$
\begin{aligned}
&\mathfrak{B}\mathrm{Y}^2 + 2\mathfrak{F}\mathrm{YZ} + \mathfrak{C}\mathrm{Z}^2\\
&= (\mathrm{CX}^2 - 2\mathrm{GZX} + \mathrm{AZ}^2)\mathrm{Y}^2 - 2(\mathrm{AYZ} - \mathrm{HZX} - \mathrm{GXY} + \mathrm{FX}^2)\mathrm{YZ} + (\mathrm{AY}^2 - 2\mathrm{HXY} + \mathrm{BX}^2)\mathrm{Z}^2\\
&= (\mathrm{CY}^2 - 2\mathrm{FYZ} + \mathrm{BZ}^2)\mathrm{X}^2\\
&= \mathfrak{A}\mathrm{X}^2
\end{aligned}
$$

also

$$\begin{aligned}&\mathfrak{C}Z+\mathfrak{F}Y\\ &=(AY^2-2HXY+BX^2)Z-(AYZ-HZX-GXY+FX^2)Y\\ &=(BZX-FXY-HYZ+GY^2)X\\ &=-\mathfrak{G}X\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{B}Y+\mathfrak{F}Z\\ &=(CX^2-2GZX+AZ^2)Y-(AYZ-HZX-GXY+FX^2)Z\\ &=(CXY-GYZ-FZX+HZ^2)X\\ &=-\mathfrak{H}X\end{aligned}$$

Hence the whole expression

$$=(\mathfrak{A}X'^2+\mathfrak{B}Y'^2+\mathfrak{C}Z'^2+2\mathfrak{F}Y'Z'+2\mathfrak{G}Z'X'+2\mathfrak{H}X'Y')X^2,$$

or

$$(\mathfrak{A},\mathfrak{B},\ldots)(X',Y',Z')^2=0$$

i.e.,

$$\begin{array}{lllll} A, & H, & G, & D_x, & D_x'=0 \\ H, & B, & F, & D_y, & D_y' \\ G, & F, & C, & D_z, & D_z' \\ D_x, & D_y, & D_z, & . & . \\ D_x', & D_y', & D_z', & . & . \end{array} \qquad \ldots\ldots\ldots\ldots \quad (21)$$

The equations (19), and consequently also the equations (19) and (20), are therefore together equivalent only to the single condition (21).

Now the relation (21) has been derived from the form T; if to this we add the corresponding relations derived from the forms U, V, W, we shall have four relations. Others are readily obtained as follows. If we form the four systems of which (17) is one, we may write them down thus:—

$$\begin{array}{lllll} & \mathfrak{B}=|\gamma,\alpha,\delta|(A,z,t), & \mathfrak{C}=|\alpha,\beta,\delta|(A,y,t), & \mathfrak{D}=|\alpha,\beta,\gamma|(A,y,z), & (\text{suffix } x)\\ \mathfrak{A}=|\beta,\gamma,\delta|(B,z,t), & . & \mathfrak{C}=|\alpha,\beta,\delta|(B,z,t), & \mathfrak{D}=|\alpha,\beta,\gamma|(B,z,x), & (\;\;,,\;\; y)\\ \mathfrak{A}=|\beta,\gamma,\delta|(C,y,t), & \mathfrak{B}=|\gamma,\alpha,\delta|(C,x,t), & . & \mathfrak{D}=|\alpha,\beta,\gamma|(C,x,y), & (\;\;,,\;\; z)\\ \mathfrak{A}=|\beta,\gamma,\delta|(D,y,z), & \mathfrak{B}=|\gamma,\alpha,\delta|(D,z,x), & \mathfrak{C}=|\alpha,\beta,\delta|(D,x,y), & . & (\;\;,,\;\; t)\end{array}$$

Whence

$$\begin{array}{llll} . & \mathfrak{A}:(B,z,t)= & \mathfrak{A}:(C,y,t)= & \mathfrak{A}:(D,y,z), \quad . \quad . \quad (22)\\ \mathfrak{B}:(A,z,t)= & . & =\mathfrak{B}:(C,x,t)= & \mathfrak{B}:(D,z,x),\\ \mathfrak{C}:(A,y,t)= & \mathfrak{C}:(B,x,t)= & . & =\mathfrak{C}:(D,x,y),\\ \mathfrak{D}:(A,y,z)= & \mathfrak{D}:(B,z,x)= & \mathfrak{D}:(C,x,y)= & .\\ (\text{suffix } x) & (\text{suffix } y) & (\text{suffix } z) & (\text{suffix } t)\end{array}$$

We thus have eight more relations. And if to these we add the corresponding relations derived from the forms U′, V′, W′, T′, we shall have 2(4+8)=24 relations in addition to (16), viz., 8+24=32 in all.]

From this we may conclude the following group,

$$\begin{aligned} 2\mathfrak{F} &= |\alpha, \beta, \delta|\,(\mathrm{D}, z, x) + |\gamma, \alpha, \delta|\,(\mathrm{D}, x, y) \quad . \quad . \quad . \quad . \quad . \quad (18) \\ 2\mathfrak{G} &= |\alpha, \beta, \delta|\,(\mathrm{D}, y, z) + |\beta, \gamma, \delta|\,(\mathrm{D}, x, y), \\ 2\mathfrak{H} &= |\gamma, \alpha, \delta|\,(\mathrm{D}, y, z) + |\beta, \gamma, \delta|\,(\mathrm{D}, z, x). \end{aligned}$$

From (17) and (18) we can now eliminate the three quantities $|\beta, \gamma, \delta|$, $|\gamma, \alpha, \delta|$, $|\alpha, \beta, \delta|$, and obtain the following identical relations,

$$\begin{aligned} &\mathfrak{B}(\mathrm{D}, x, y)^2 - 2\mathfrak{F}(\mathrm{D}, x, y)(\mathrm{D}, z, x) + \mathfrak{C}(\mathrm{D}, z, x)^2 = 0 \quad . \quad . \quad . \quad . \quad (19) \\ &\mathfrak{C}(\mathrm{D}, y, z)^2 - 2\mathfrak{G}(\mathrm{D}, y, z)(\mathrm{D}, x, y) + \mathfrak{A}(\mathrm{D}, x, y)^2 = 0 \\ &\mathfrak{A}(\mathrm{D}, z, x)^2 - 2\mathfrak{H}(\mathrm{D}, z, x)(\mathrm{D}, y, z) + \mathfrak{B}(\mathrm{D}, y, z)^2 = 0 \end{aligned}$$

to these may be added

$$\begin{aligned} &-\mathfrak{F}(\mathrm{D}, y, z)^2 - \mathfrak{A}(\mathrm{D}, z, x)(\mathrm{D}, x, y) + \mathfrak{H}(\mathrm{D}, x, y)(\mathrm{D}, y, z) + \mathfrak{G}(\mathrm{D}, y, z)(\mathrm{D}, z, x) = 0 \quad . \quad (20) \\ &-\mathfrak{G}(\mathrm{D}, z, x)^2 + \mathfrak{H}(\mathrm{D}, z, x)(\mathrm{D}, x, y) - \mathfrak{B}(\mathrm{D}, x, y)(\mathrm{D}, y, z) + \mathfrak{F}(\mathrm{D}, y, z)(\mathrm{D}, z, x) = 0 \\ &-\mathfrak{H}(\mathrm{D}, x, y)^2 + \mathfrak{G}(\mathrm{D}, z, x)(\mathrm{D}, x, y) + \mathfrak{F}(\mathrm{D}, x, y)(\mathrm{D}, y, z) - \mathfrak{C}(\mathrm{D}, y, z)(\mathrm{D}, z, x) = 0 \end{aligned}$$

But, as the six equations (19) and (20) are in any case equivalent to only three independent conditions, it is not necessary to go beyond the equations (19).

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Added April 23, 1881.

In the present case however the equations (19) are themselves not independent, as may be shown in the following way. Write for the moment

$$\mathrm{D}_x = \mathrm{X},\ \mathrm{D}_y = \mathrm{Y},\ \mathrm{D}_z = \mathrm{Z};\ \mathrm{D}_x' = \mathrm{X}',\ \mathrm{D}_y' = \mathrm{Y}',\ \mathrm{D}_z' = \mathrm{Z}';$$

then

$$(\mathrm{D}, y, z) = \mathrm{Y}'\mathrm{Z} - \mathrm{Y}\mathrm{Z}',\ (\mathrm{D}, z, x) = \mathrm{Z}'\mathrm{X} - \mathrm{Z}\mathrm{X}',\ (\mathrm{D}, x, y) = \mathrm{X}'\mathrm{Y} - \mathrm{X}\mathrm{Y}'.$$

Inserting these values in the first equations of (19) we obtain

$$\begin{aligned} &\mathfrak{B}(\mathrm{X}'\mathrm{Y} - \mathrm{X}\mathrm{Y}')^2 - 2\mathfrak{F}(\mathrm{X}'\mathrm{Y} - \mathrm{X}\mathrm{Y}')(\mathrm{Z}'\mathrm{X} - \mathrm{Z}\mathrm{X}') + \mathfrak{C}(\mathrm{Z}'\mathrm{X} - \mathrm{Z}\mathrm{X}')^2 = 0 \\ &\quad = (\mathfrak{B}\mathrm{Y}^2 + 2\mathfrak{F}\mathrm{YZ} + \mathfrak{C}\mathrm{Z}^2)\mathrm{X}'^2 + \mathfrak{B}\mathrm{X}^2.\mathrm{Y}'^2 + \mathfrak{C}\mathrm{X}^2.\mathrm{Z}'^2 + 2\mathfrak{F}\mathrm{X}^2.\mathrm{Y}'\mathrm{Z}' \\ &\quad - 2\mathfrak{C}\mathrm{ZX}.\mathrm{Z}'\mathrm{X}' - 2\mathfrak{B}\mathrm{XY}.\mathrm{X}'\mathrm{Y}' - 2\mathfrak{F}\mathrm{XY}.\mathrm{Z}'\mathrm{X}' - 2\mathfrak{F}\mathrm{ZX}.\mathrm{X}'\mathrm{Y}'. \end{aligned}$$

But

$$\begin{aligned} &\mathfrak{B}\mathrm{Y}^2 + 2\mathfrak{F}\mathrm{YZ} + \mathfrak{C}\mathrm{Z}^2 \\ &= (\mathrm{CX}^2 - 2\mathrm{GZX} + \mathrm{AZ}^2)\mathrm{Y}^2 - 2(\mathrm{AYZ} - \mathrm{HZX} - \mathrm{GXY} + \mathrm{FX}^2)\mathrm{YZ} + (\mathrm{AY}^2 - 2\mathrm{HXY} + \mathrm{BX}^2)\mathrm{Z}^2 \\ &= (\mathrm{CY}^2 - 2\mathrm{FYZ} + \mathrm{BZ}^2)\mathrm{X}^2 \\ &= \mathfrak{A}\mathrm{X}^2 \end{aligned}$$

also

$$\begin{aligned}&\mathfrak{C}\mathrm{Z}+\mathfrak{F}\mathrm{Y}\\ &=(\mathrm{AY}^2-2\mathrm{HXY}+\mathrm{BX}^2)\mathrm{Z}-(\mathrm{AYZ}-\mathrm{HZX}-\mathrm{GXY}+\mathrm{FX}^2)\mathrm{Y}\\ &=(\mathrm{BZX}-\mathrm{FXY}-\mathrm{HYZ}+\mathrm{GY}^2)\mathrm{X}\\ &=-\mathfrak{G}\mathrm{X}\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{B}\mathrm{Y}+\mathfrak{F}\mathrm{Z}\\ &=(\mathrm{CX}^2-2\mathrm{GZX}+\mathrm{AZ}^2)\mathrm{Y}-(\mathrm{AYZ}-\mathrm{HZX}-\mathrm{GXY}+\mathrm{FX}^2)\mathrm{Z}\\ &=(\mathrm{CXY}-\mathrm{GYZ}-\mathrm{FZX}+\mathrm{HZ}^2)\mathrm{X}\\ &=-\mathfrak{H}\mathrm{X}\end{aligned}$$

Hence the whole expression

$$=(\mathfrak{A}\mathrm{X}'^2+\mathfrak{B}\mathrm{Y}'^2+\mathfrak{C}\mathrm{Z}'^2+2\mathfrak{F}\mathrm{Y}'\mathrm{Z}'+2\mathfrak{G}\mathrm{Z}'\mathrm{X}'+2\mathfrak{H}\mathrm{X}'\mathrm{Y}')\mathrm{X}^2,$$

or

$$(\mathfrak{A},\ \mathfrak{B},\ \ldots)(\mathrm{X}',\ \mathrm{Y}',\ \mathrm{Z}')^2=0$$

i.e.,

$$\begin{array}{lllll}\mathrm{A}, & \mathrm{H}, & \mathrm{G}, & \mathrm{D}_x, & \mathrm{D}_x'=0\\ \mathrm{H}, & \mathrm{B}, & \mathrm{F}, & \mathrm{D}_y, & \mathrm{D}_y'\\ \mathrm{G}, & \mathrm{F}, & \mathrm{C}, & \mathrm{D}_z, & \mathrm{D}_z'\\ \mathrm{D}_x, & \mathrm{D}_y, & \mathrm{D}_z, & . & .\\ \mathrm{D}_x', & \mathrm{D}_y', & \mathrm{D}_z', & . & .\end{array} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The equations (19), and consequently also the equations (19) and (20), are therefore together equivalent only to the single condition (21).

Now the relation (21) has been derived from the form T; if to this we add the corresponding relations derived from the forms U, V, W, we shall have four relations. Others are readily obtained as follows. If we form the four systems of which (17) is one, we may write them down thus :—

$$\begin{array}{lllll} . & \mathfrak{B}=|\gamma,\alpha,\delta|(\mathrm{A},z,t), & \mathfrak{C}=|\alpha,\beta,\delta|(\mathrm{A},y,t), & \mathfrak{D}=|\alpha,\beta,\gamma|(\mathrm{A},y,z), & (\text{suffix } x)\\ \mathfrak{A}=|\beta,\gamma,\delta|(\mathrm{B},z,t), & . & \mathfrak{C}=|\alpha,\beta,\delta|(\mathrm{B},z,t), & \mathfrak{D}=|\alpha,\beta,\gamma|(\mathrm{B},z,x), & (\ \ ,,\ \ y)\\ \mathfrak{A}=|\beta,\gamma,\delta|(\mathrm{C},y,t), & \mathfrak{B}=|\gamma,\alpha,\delta|(\mathrm{C},x,t), & . & \mathfrak{D}=|\alpha,\beta,\gamma|(\mathrm{C},x,y), & (\ \ ,,\ \ z)\\ \mathfrak{A}=|\beta,\gamma,\delta|(\mathrm{D},y,z), & \mathfrak{B}=|\gamma,\alpha,\delta|(\mathrm{D},z,x), & \mathfrak{C}=|\alpha,\beta,\delta|(\mathrm{D},x,y), & . & (\ \ ,,\ \ t)\end{array}$$

Whence

$$\begin{array}{llll} . & \mathfrak{A}:(\mathrm{B},z,t)= & \mathfrak{A}:(\mathrm{C},y,t)= & \mathfrak{A}:(\mathrm{D},y,z), \quad . \quad . \quad (22)\\ \mathfrak{B}:(\mathrm{A},z,t)= & . & =\mathfrak{B}:(\mathrm{C},x,t)= & \mathfrak{B}:(\mathrm{D},z,x),\\ \mathfrak{C}:(\mathrm{A},y,t)= & \mathfrak{C}:(\mathrm{B},x,t)= & . & =\mathfrak{C}:(\mathrm{D},x,y),\\ \mathfrak{D}:(\mathrm{A},y,z)= & \mathfrak{D}:(\mathrm{B},z,x)= & \mathfrak{D}:(\mathrm{C},x,y)= & .\\ (\text{suffix } x) & (\text{suffix } y) & (\text{suffix } z) & (\text{suffix } t)\end{array}$$

We thus have eight more relations. And if to these we add the corresponding relations derived from the forms U′, V′, W′, T′, we shall have 2(4+8)=24 relations in addition to (16), viz., 8+24=32 in all.]

To obtain two more, write for the moment

$$|\beta, \gamma, \delta| = p^{-1}, \quad |\gamma, \alpha, \delta| = q^{-1}, \quad |\alpha, \beta, \delta| = r^{-1}, \quad |\alpha, \beta, \gamma| = s^{-1} \quad . \quad (23)$$

and then multiply (17) by pD_x, qD_y, rD_z respectively and add them together. This will give

$$\mathfrak{A}pD_x + \mathfrak{B}qD_y + \mathfrak{C}rD_z = 0.$$

And if we proceed in a similar way with the corresponding systems derived from U, V, W respectively, we may form the following system

$$\begin{array}{llll} \quad . \quad \mathfrak{B}A_y q + \mathfrak{C}A_z r - \mathfrak{D}A_t s = 0, & \mathfrak{A}, \mathfrak{B}, \ldots & \text{suffix } x & \quad . \quad . \quad . \quad (24) \\ \mathfrak{A}B_x p + \quad . \quad + \mathfrak{C}B_z r - \mathfrak{D}B_t s = 0, & \quad ,, & \quad ,, \quad y, & \\ \mathfrak{A}C_x p + \mathfrak{B}C_y q + \quad . \quad - \mathfrak{D}C_t s = 0, & \quad ,, & \quad ,, \quad z, & \\ \mathfrak{A}D_x p + \mathfrak{B}D_y q + \mathfrak{C}D_z r + \quad . \quad = 0, & \quad ,, & \quad ,, \quad t, & \end{array}$$

whence eliminating p, q, r, s, we finally obtain the following relation,

$$\begin{array}{llll} \quad . \quad \mathfrak{B}A_y, \ \mathfrak{C}A_z, \ \mathfrak{D}A_t = 0, & \mathfrak{A}, \mathfrak{B}, \ldots & \text{suffix } x & . \quad . \quad . \quad . \quad . \quad (25) \\ \mathfrak{A}B_x, \quad . \quad \mathfrak{C}B_z, \ \mathfrak{D}B_t & \quad ,, & \quad ,, \quad y & \\ \mathfrak{A}C_x, \ \mathfrak{B}C_y, \quad . \quad \mathfrak{D}C_t & \quad ,, & \quad ,, \quad z & \\ \mathfrak{A}D_x, \ \mathfrak{B}D_y, \ \mathfrak{C}D_z, \quad . & \quad ,, & \quad ,, \quad t & \end{array}$$

To this may be added the corresponding relation obtained from the forms U′, V′, W′, T′. These added to the former conditions give us $32+2=34$.

It was however remarked at the outset that the equations $UP'-U'P=0$, &c., are lineo-linear in the U coordinates, and also in the U′ coordinates; and as we are concerned with the ratios only of the coefficients, and not with their absolute values, we are in fact concerned only with the ratios of the U coordinates *inter se*, and with those of the U ′coordinates *inter se*, and not with their absolute values. Hence the number of independent coordinates will finally be reduced to

$$48-34-2=12,$$

as it should be.

VII. *On the Viscosity of Gases at High Exhaustions.*

By WILLIAM CROOKES, *F.R.S.*

Received December 26, 1880,—Read February 17, 1881.

[PLATES 55–58.]

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INTRODUCTION.

635.* By the viscosity or internal friction of a gas, is meant the resistance it offers to the gliding of one portion over another. In a paper read before the British Association in 1859, MAXWELL† gives the following explanation of the internal friction of gases:—

"Particles having the mean velocity of translation belonging to one layer of the gas, pass out of it into another layer having a different velocity of translation, and by striking against the particles of the second layer exert upon it a tangential force which constitutes the internal friction of the gas. The whole friction between two portions of gas separated by a plane surface, depends upon the total action between all the layers on the one side of that surface upon all the layers on the other side."

The research here recorded commenced with experiments instituted to discriminate between the friction of the pivot supporting the fly of a radiometer and the viscosity of the residual gas. In the paper just quoted, MAXWELL presented the remarkable result that on theoretical grounds the coefficient of friction, or the viscosity, should be independent of the density of the gas, although at the same time he states that the only experiments he had met with on the subject did not seem to confirm his views.

An elaborate series of experiments were undertaken by MAXWELL to test so remarkable a consequence of a mathematical theory; and in 1866, in the Bakerian lecture for that year,‡ he published the results under the title of "The Viscosity or Internal Friction of Air and other Gases." He found the coefficient of friction in air to be practically constant for pressures between 30 inches and 0·5 inch; in fact, numbers calculated on the hypothesis that the viscosity was independent of the density agreed very well with the observed values.

The apparatus used by MAXWELL was not of a character to admit of experiments with much lower pressures than 0·5 inch.

636. In the Philosophical Magazine for July, 1875, is a translation of a paper by MM. KUNDT and WARBURG, "On Friction and Heat Conduction in Rarefied Gases," in which the laws theoretically discovered by MAXWELL were examined at higher exhaustions.

MAXWELL'S theory, that the viscosity of a gas is independent of the density, presupposes that the mean length of path of the molecules between their collisions is very small compared with the dimensions of the apparatus; but inasmuch as the mean length of path increases directly with the expansion, whilst the distance between the molecules only increases with the cube root of the expansion, it is not difficult with the SPRENGEL pump to produce an exhaustion in which the mean free path is measured by inches, and even feet,* and at exhaustions of this degree it is probable that MAXWELL's law would not hold good.

---

* The research embodied in this paper is an outcome of work recorded in my various papers "On Repulsion resulting from Radiation," Parts 1 to 6, "On the Illumination of Lines of Molecular Pressure, and the Trajectory of Molecules," and "Contributions to Molecular Physics in High Vacua." The paragraphs are therefore numbered in continuation of the last-named paper.

† Phil. Mag., 4th ser., vol. xix., p. 31.

‡ Phil. Trans., 1866, Part I., p. 249.

MM. KUNDT and WARBURG found that for pressures between 760 millims. and 1·5 millim. the coefficient of viscosity of air was constant, but at higher vacua it fell off. They, however, give no measurements of the amount of exhaustion obtained, simply speaking of Vacua I., II., III., and IV.

637. As I have had considerable experience in working in high vacua, and am accustomed to measure with accuracy exhaustions up to the ten-millionth of an atmosphere and even higher, it has been proposed that I should continue these experiments on viscosity of gases at high exhaustions, at the same time obtaining as many other data and measurements as the apparatus can be made to afford.

My experiments were commenced early in 1876, and have been continued to the present time. In November, 1876, I gave a note to the Royal Society on some preliminary results. Several different forms of apparatus have since been used one after the other, with improvements and complexities suggested by experience or rendered possible by the extra skill acquired in manipulation. The earlier observations are now of little value, but the time spent in their prosecution was not thrown away, as out of those experiments has grown the very complicated apparatus now finally adopted. I will therefore not occupy time in describing earlier forms of apparatus, but will proceed at once to the one finally adopted.

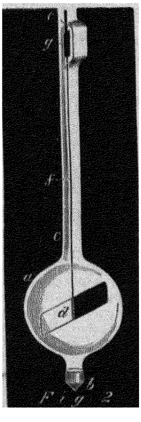
THE VISCOSITY TORSION APPARATUS.

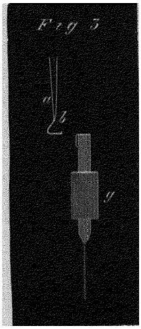
638. Plate 55, fig. 1, shows the general construction, fig. 2 (p. 390) an enlarged view of the torsion beam, &c., the same references applying to either figure. *a* is a glass bulb, blown with a point at the lower end, *b*, and sealed on to a long narrow glass tube *c c*. In the bulb is suspended a plate of mica, *d*, by means of a fine fibre of glass, 26 inches long, which is sealed to the top of the glass tube *c c*, and hangs vertically along its axis. The plate of mica is ignited and lamp-blacked over one-half, as shown. The tube *c c* is pointed at the upper end, *e*; the points *e* and *b* are 46 inches apart, and are accurately in the prolongation of the axis of the tube. Sockets are firmly fixed at *b* and *e* to a solid support, so that when the tube and bulb are clamped between them

---

* Thus, supposing the mean free path of the molecules of air at the ordinary pressure is the $\frac{1}{10000}$th of a millimetre, at an exhaustion of the ten-thousandth of an atmosphere, the mean free path will be 1 millim. At one-millionth of an atmosphere the mean free path will be 10 centims., and at an exhaustion of one hundred millionth—by no means a difficult point to attain with present appliances—the mean free path will be over 30 feet. This rarefaction corresponds to that of the atmosphere at a height above the earth of a little more than 90 miles, assuming that its density decreases in geometrical progression as its height increases in arithmetical progression, and neglecting the small corrections for diminished gravity and temperature. As the height above the earth increases, the length of the mean free path of the molecules of air rapidly approaches to planetary distances; at about 200 miles height the mean free path is 10,000,000 miles, whilst between 80 and 90 miles higher the rarity is such that the mean free path would extend from here to Sirius.

they are only able to move around the vertical axis. The upper socket is lined with cork and is pressed down by a spring. The lower socket is a steel cup, and a brass point is attached to the bottom of the bulb. This device is adopted because in the earlier apparatus the friction of glass against cork generated electricity, which interfered with the free movement of the torsion fibre. The glass fibre being only connected with the tube at *e*, rotating the tube on its axis communicates torsion to the fibre and sets the mica plate swinging on the same axis without giving it any pendulous movement. The mica plate *d* is not fastened direct to the suspending fibre, but through the intermediary of a twisted aluminium wire, *f*, about 12 inches long, with a mirror of platinised glass attached to it. The tube opposite the mirror is blown into the form of a thin cylinder of about double its original diameter, and the centre of curvature is made to coincide as closely as possible with the fibre, or centre of motion of the mirror. There are three reasons why the mirror is kept some distance above the mica plate, instead of being close to it:—1. There is much less distortion of the reflected index of light when passing through a cylinder than through a bulb. 2. The light falling on the mirror from the lamp might diffuse sufficiently to produce repulsion of the black surface. 3. The torsion bulb and plate can be completely packed in wool or immersed in water, to diminish variation of temperature.





The total weight of mica plate, mirror, and aluminium support is 5·69 grains. These are purposely made very light, and to render the effects of viscosity more apparent the glass fibre is as fine as it can be drawn consistent with safety. The diameter of the fibre is about 0·001 inch. The attachment of so thin a fibre at each end of the apparatus requires extreme care, or it will snap off at a joint when torsion is applied. The following plan has been found to answer best:—The fibre being drawn from a piece of glass rod about 0·5 millim. in diameter is left attached at one end to the rod from which it was drawn, the other end being broken off to the right length. This broken end of the fine torsion fibre is inserted into the wider end of a fine, somewhat conical, glass tube *a*, fig. 3 (which must not be much larger than is necessary to allow the fibre to enter freely). A small flame applied at *b* melts the glass tube and fibre together. The lower part of the glass tube is bent, as shown in the figure, to allow the aluminium support for the mirror, *g*, and mica plate to be hooked on to it. The small piece of glass rod remaining attached to the upper part of the torsion fibre is then passed through, so as to project above the top of the glass tube *c c*, fig. 1, which has previously been drawn out to the right size, and held in position by an aluminium spring. When the instrument is held in a vertical position, the final adjustment can be made by moving the projecting end of the rod; the top aperture is then sealed, and the piece of rod carrying the fibre is likewise sealed in at the same time. The viscosity apparatus is connected to the pump by the arm *h*, and tubes *i*, *j*, *k*, *l*, *m*, *n*, and *o*. A flexible glass spiral is introduced at *i*, so as to allow the apparatus to rotate on the pivots *b*, *e*, and at the same time to be connected to the pump altogether with sealed glass joints. Another arm at *p*, working between metal stops, limits the rotation to the small angle only which is necessary. A spring keeps the arm pressed close to one of these stops.

639. The bulb of the viscosity apparatus is entirely enclosed in a box, packed with cotton-wool; in this box is a tube, *q*, opposite the black half of the mica plate, so that a candle at *r* will shine on it and produce a deflection. This tube is closed at each end with glass plates to keep out air-currents, and in front is a shutter to cut off or turn on the light of the candle, as desired. The position of the candle flame and its distance from the plate are always kept identical in the following manner:—A small lens, fixed at some distance from the candle, forms an image of the flame on a white screen, and it is so adjusted that when the flame shines direct on to the mica plate its image exactly overlaps a mark on the screen. A second system of lens and screen is fixed at right angles to the first, and thus the position of the flame can in a few moments be adjusted. To avoid unnecessary complication this arrangement is not shown in the figure.

In viscosity experiments when the deflection produced by the candle-flame is not measured—as, for instance, in exhaustions below one millimetre—the whole apparatus is immersed in a large copper vessel full of water surrounded by a double row of Winchester quart bottles, also full of water, and the spaces between them and the copper vessel are filled with cotton-wool. A cardboard case surrounds and covers the whole. A thermometer reading to tenths of a degree C. is in the water close to the bulb. This device, which I have long adopted, and found extremely useful, prevents the temperature of the apparatus during a long day's work from varying more than the tenth of a degree.

The torsional movement given to the mica plate, by the light of the candle shining on it or by the rotation of the bulb and tube on its axis by the movement of the arm *p* between the stops, is measured by a beam of light from the lamp *s*, reflected from the mirror *g* to the graduated scale *t*.

640. The angle through which the apparatus is rotated is small. The distance of the scale from the mirror is 4 feet, and the amplitude of swing carries the line of light between 100 and 200 divisions of the scale. 100 divisions of the scale $=62$ millims., so that the deflection of the line of light is never more than about 10 degrees. As the scale forms the tangent of a circle a small correction is required for the scale-reading. Let S be the scale-reading; R the radius or distance of mirror (centre of motion) from the scale; S′ what would be the scale-reading if it were curved in a circle of radius, R; then—

$$S' = S\left\{1-\tfrac{1}{3}\left(\frac{S}{R}\right)^2+\tfrac{1}{5}\left(\frac{S}{R}\right)^4-\&c.\right\}$$

The ratio of S to R is, however, so small as to make $\frac{1}{5}\frac{S^5}{R^4}$ and the succeeding terms insensible. The correction therefore is simply—

$$-\tfrac{1}{3}\frac{S^3}{R^2}.$$

Except in special cases this correction was not applied, for unless the amplitudes were very long the amount to be deducted was less than the average error of each observation.

641. Air is introduced into the apparatus through a very fine glass tube drawn out to almost the fineness of a hair, and projecting from the connecting tubes at A. A momentary contact with a spirit flame seals up the point, and when air is to be admitted the tip of the tube is broken off, and the air slowly enters, passing through the drying-tubes, *m*, before it reaches the apparatus.

Gases other than atmospheric air are introduced into the apparatus through the tubes *u* and *v*. The tubes *u* and *v* are only open when actually in use, and when the apparatus has been filled with a pure gas and exhausted a sufficient number of times, they are sealed off at *u* or at *v*, as the case may be, and when required are again attached for another gas. This prevents the apparatus when actually at work having any mercury joints, blown joints being used in all other cases. At *m* is a system of six tubes, for the double object of drying the gas and of keeping mercury vapour out of the torsion bulb. The tube nearest the pump contains pure sulphur which has been fused, then finely powdered, and introduced while warm into the tube. The next tube contains finely-divided copper reduced by hydrogen from the oxide, and the four remaining tubes contain phosphoric anhydride. The phosphoric anhydride must be tightly packed in the tubes, so as to force the air to pass through, and not merely over it, or the last traces of water will not be removed. The sulphur tube stops mercury vapour, and in the course of some months becomes darkened at the end nearest the pump. A tube packed with gold-leaf was used in the early experiments, but as it did not entirely stop the passage of mercury vapour, it was discarded in favour of sulphur, which, whilst not quite perfect, answers fairly well. As it was thought possible that sulphur vapour might be given off, the copper tube was interposed: at first a tube of oxide of lead was used for this purpose, but copper seems to answer better. At *w* a similar system of six tubes is interposed in the path of the entering gas. These will be further alluded to when the experiments with gases other than air are described. The tube *o* is the phosphoric anhydride reservoir belonging to the pump; *x* is the McLeod measuring apparatus; *y* is a small radiometer, having a standard candle at a fixed distance from it; *k* is a spectrum tube with capillary bore furnished with aluminium poles; *l* is a resistance tube having aluminium terminals with ends 4 millims. apart (663, 664, 665). At *z* a series of spectrum tubes is attached, so that in course of experiment one tube can be sealed off at any particular pressure for further reference.

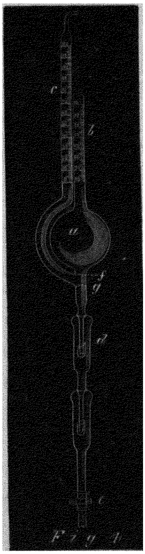


Fig. 4

642. The pump employed has already been described (355). The measuring apparatus is similar to that described by Professor McLeod* before the Physical Society, June 13th, 1874. As it contains several improvements shown by experience to be necessary when working at very high vacua, a detailed description may be useful. The instrument shown in fig. 4 consists essentially of a globe, *a*, a volume tube, *b*,

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* Philosophical Magazine, vol. xlviii., p. 110, August, 1874.

and a pressure tube, *c*. The upper part of *c* communicates with the exhaust arm of the pump, and the tube *e* communicates with the upper mercury reservoir by means of a screw tap. The pressure and volume tubes are from the same piece, accurately of the same internal diameter, and are graduated in millimetres. The divisions in the pressure tube are numbered from below upwards, and extend to some height above the volume tube; and the divisions of the volume tube are numbered from above downwards, the lowest division, 80, being on a level with the 0 of the pressure tube. The proportion between the contents of the volume tube and the globe was ascertained in the manner directed by Professor McLeod, and it was found that the globe, &c., from the point *f* had 111·8 times the capacity of the volume tube down to the 80th division. The action of the instrument is as follows:—Before the viscosity apparatus on the pump is exhausted the tap *e* is closed, in order to exclude mercury from the measuring instrument, the upper part being in communication with the pump and viscosity apparatus. The pump is now set to work, and the exhaustion in the globe keeps pace with that in the apparatus, there being free passage from one to the other. When the exhaustion is good it is advisable carefully to unscrew the tap *e* a little, to allow mercury to fill the two air-traps, *d d*, and rise in the tube as far as *g*. The tap *e* is then tightly closed, and the exhaustion continued. When it is desired to make a measurement of the pressure, the tap *e* is opened, and the mercury allowed slowly to rise. When it reaches the point *f* it cuts off communication between the globe *a* and the rest of the apparatus. The globe *a* now contains an accurately measured volume of highly rarefied gas of exactly the same exhaustion as that in the viscosity apparatus. The mercury continuing to rise, part ascends the pressure tube *c*, and part fills the globe, compressing the rarefied gas, until, when it reaches the lowest division, 80, of the volume tube, the gas is condensed to 111·8 times its original volume. I now quote from Professor McLeod's description:—"Ultimately the whole of the gas in the globe is condensed into the volume tube; and its tension is then found by measuring the difference of level between the columns of mercury in the volume and pressure tubes. On dividing this difference by the ratio between the capacities of the globe and volume tube, a number is obtained which is approximately the original pressure of the gas: this number must now be added to the difference between the columns, since it is obvious that the column in the pressure tube is depressed by the tension of the gas in the remaining part of the apparatus: on dividing this new number once more by the ratio between the volumes, the exact original tension is found. . . . The relations existing between the contents of the other divisions of the volume tube and the total contents of the globe were determined by measuring the tensions of the same quantity of gas when compressed into the different volumes." In this way a table has been constructed giving the value of each millimetre in the volume tube from 1 to 80. I generally make several readings at different heights of the volume tube, and take the mean. The McLeod gauge will not show the presence of mercury vapour. I have, however, entirely failed to detect the presence of mercury vapour at any great distance from the mercury in the pump; and the tube packed with gold-leaf, which I formerly interposed between the pump and the apparatus, showed no trace of bleaching, and exerted no appreciable effect one way or the other on the results. The chief improvements in this instrument over Professor McLeod's original design consist in the entire absence of joints; the tubes and bulbs are all soldered together in one piece; and the two air-traps, *d d*, which obviate the inconvenience at first encountered, of traces of air rising with the mercury and spoiling the vacuum in *a*. The globe *a* is also larger in proportion to the volume tube, and this tube is longer. These improvements are mere details, and in no way detract from the great beauty and merit of Professor McLeod's valuable instrument.

643. In the case of a body moving quickly through air of ordinary density the "work" is almost wholly represented by a wake of eddies left behind. The smaller the velocity of the body and the rarer the gas the less is the work spent in producing eddies compared with the work spent in *directly* overcoming friction. In these experiments the motion is so slow that even at full pressure the effect of eddies is well nigh or altogether invisible, and it is utterly insensible at even moderate exhaustions. The molar *vis viva* is rapidly converted by internal friction or viscosity into molecular *vis viva* (heat, thermometric—not radiant), and this is too rapidly dissipated to interfere with the observations.

644. Before taking an observation the arm *p* (fig. 1) remains pressed against one of the stops. In this position the index line of light stands in the middle of the scale at 0, the divisions counting on each side from 0 upwards. The arm is then moved over to the other stop, and in a few seconds allowed to return to its original position by the action of the spring. This movement rotates the viscosity apparatus through a small angle, and sets the mica plate vibrating, the reflected line of light traversing from one side of the scale to the other in arcs of diminishing amplitude till it finally settles down once more at *zero*. The amplitude depends on the manipulation, a slower turn being more effective in producing motion than a very quick one. Even if the arm *p* is turned at the same rate, in experiments with different degrees of exhaustion there will be no simple relation between the arc swung through and the viscosity. What is simply related to the viscosity is the logarithmic decrement of the arc of oscillation. The logarithmic decrement will of course involve the viscosity of the glass; but glass is so nearly perfectly elastic, and the fibre is so very thin, that this will be practically insensible except at the very highest exhaustions (652).

The observer watching the moving index of light records the scale number reached at the extremity of each arc. The numbers being alternately on one and the other side of zero are added two by two together, to get the value of each oscillation. The logarithms of these values are then found, and their differences taken; the mean of these differences is the logarithmic decrement per swing of the arc of oscillation.

The following illustration will render this plain. The observation is supposed to be taken with air at the pressure of 760 millims. :—

Readings.	Arcs.	Log. arcs.	Diff.
113·0			
	200·0	2·3010	
87·0			0·1124
	154·4	2·1886	
67·4			0·1123
	119·2	2·0763	
51·8			0·1116
	92·0	1·9647	
40·2			0·1134
	71·0	1·8513	
30·8			0·1125
	54·8	1·7388	
24·0			0·1125
	42·3	1·6263	
18·3			
	Mean		0·1124

The logarithmic decrement for the oscillation comes out 0·1124.

645. The actual number obtained varies with each apparatus. The number does not represent an absolute, but a relative, quantity, to be compared only with the results obtained with the same apparatus at other pressures or with other gases.

The first swing is seldom found to be in regular series. It is better to start an oscillation greater than is intended to be observed, so that the oscillation shall be fairly established when beginning to preserve the scale-readings.

On comparing a large number of observations there appears to be a tendency for the logarithmic decrements to come alternately greater and less than the mean. It is quite conceivable that this may be real, and not accidental, as the body is moving opposite ways—just as the resistance to a ship would be different according as she went prow foremost or stern foremost. Therefore it is well to take an *odd* number of arcs for final discussion, leaving an *even* number of intervals, so that the difference, if any, between fore and aft motion should be eliminated.

646. Instead of taking the whole number of arcs it comes to the same thing to use the initial and final arc only, except for the sake of comparing the individual results with the mean of the whole. To take a particular instance,—say there are 7 arcs, a_1, a_2, to a_7, The mean logarithmic decrement is—

$$\tfrac{1}{6}\{(\log a_1-\log a_2)+(\log a_2-\log a_3)+(\log a_3-\log a_4)+(\log a_4-\log a_5)+ \\ +(\log a_5-\log a_6)+(\log a_6-\log a_7)\}$$

which is the same thing as $\tfrac{1}{6}(\log a_1-\log a_7)$, and the same is evidently true whatever be the number of arcs.

For instance, in the illustration already given much time will be saved by taking the observation thus :—

Readings.	Arcs.	Log. arcs.
113·0		
	200·0	2·3010
87·0		
..		
..		
..		
..		
24·0		
	42·3	1·6263
18·3		
		6)0·6747
		0·1124

647. In deducing the logarithmic decrement from the initial and final arcs it is advisable not to wait until the final arc has fallen very low, for then an error in the reading of the arc tells too much. In taking a long series a good plan is to group them into intervals of, say, 4 arcs, and take the logarithmic decrements for comparison with one another. Thus, let a_1, a_2, a_3. . . . be the arcs; we may then take

$$\begin{gathered}\tfrac{1}{4}\,(\log a_1 - \log a_5)\\ \tfrac{1}{4}\,(\log a_5 - \log a_9)\\ \tfrac{1}{4}\,(\log a_9 - \log a_{13})\\ \tfrac{1}{4}\,(\log a_{13} - \log a_{17}),\end{gathered}$$

and compare them with each other. They will be more regular than the logarithmic decrements for single intervals only, as the errors of observation will be divided by 4.

648. The *proportional* error of observation of a small arc is of course much greater than that of a large one, but the *absolute* error, if anything, is rather less. Assuming the absolute errors to be the same in each case, the best arc to stop with so that a given error in the observation of the small arc shall produce a minimum error in the deduced logarithmic decrement, we find it (the last arc) should be $\frac{1}{e}$th part of the first arc used in the computation, e being the base of the Napierian logarithms, namely, 2·71828 . . . . As the small arcs can be observed a little better than the large arcs, on account of the slowness of the motion, we may go a little lower, say to $\frac{1}{3}$rd of the first arc preserved for calculation.

649. The slight diminution in the log. dec. of air between pressures of 760 millims. and 1 millim. is clearly shown in figs. 5, 6, 7, and 8. The curved lines are copied from photographic traces made on a sensitive surface by the index ray of light.

The experiments in which photography was employed will be described later on.

The successive diminutions of arc are caused by the resistance or the viscosity of the air. Fig. 5 shows the damping action at 760 millims., and fig. 7 that at 1 millim. Figs. 6 and 8 show the diminution in length of the successive arcs at the same two pressures, the line A B joining the extremities being the logarithmic curve.

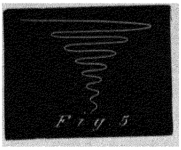
Fig. 5

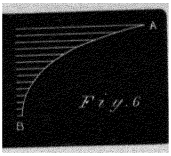

Fig. 6

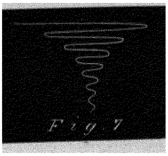
Fig. 7

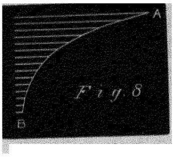

Fig. 8

In figs. 9 and 10 I give the curves representing the successive diminutions of the arc of vibration at high exhaustions due to viscosity. Fig. 9 is copied from a photograph, and shows the damping action at 0·02 M. The diminution in length of the arcs and the logarithmic curve—now nearly straight—are given in fig. 10. It is interesting to compare these figures with figs. 5, 6, 7, and 8, which show similar results at low exhaustions.

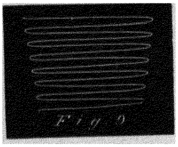
Fig. 9

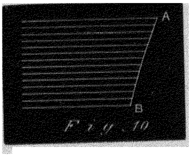

Fig. 10

A very large number of experiments have been made on the viscosity of air and other gases. To reduce the results within convenient limits I will first take the case of dry air between the ordinary atmospheric pressure and the $\frac{1}{1000}$th of an atmosphere, and will then describe the results obtained between the $\frac{1}{1000}$th and the $\frac{1}{1,000,000}$th of an atmosphere, and higher. I have taken observations at as high an exhaustion as 0·02 M, but at these high points they are not sufficiently concordant to be trustworthy. The pump will exhaust to this point without difficulty if a few precautions are taken, but at this low pressure the means of measuring fail in accuracy.

650. It may be useful if I here give some of the precautions which experience shows to be necessary when exhausting to the highest points. In the first place, the mercury must be pure. If it contains metals more easily oxidisable than itself, there is a tendency, if acids, or even water, come in contact with the mercury, for hydrogen to be liberated. This hydrogen is in part absorbed by the mercury, and, carried by it through the pump, is slowly liberated at low pressure. The consequence is that the highest exhaustions cannot be obtained, whilst the results are due to hydrogen rather

than to air. Before this discovery many months were wasted in taking useless observations, owing to the presence of hydrogen as impurity, and several pieces of viscosity apparatus and pumps were taken to pieces to find out a supposed leakage.

The best way to purify mercury is to violently agitate it with a solution of perchloride of iron. The mercury is thus rapidly converted into a pasty mass of metallic globules in so fine a state of division as to be invisible to the naked eye. The globules are washed, and then dried by squeezing first through a cloth, then through chamois leather. In these operations the minute globules coalesce, and the metal reassumes its fluidity.

651. When the high exhaustions are first approached in the apparatus another precaution is necessary. The pump is kept going till an exhaustion of about 0·5 M is attained, and the whole apparatus, including the connecting-tubes and those belonging to the pump, are heated to between 300° and 400°, by passing over them the flame of a BUNSEN burner. The apparatus itself is then kept at a temperature of about 300° for half an hour or more, and the pump is worked the whole time. Glass condenses on its surface a certain amount of permanent gas which is let out very slowly and incompletely in a vacuum; but the gas is driven off, in quantity sufficient to depress the gauge, when the temperature of apparatus and pump is quickly raised to 300° or 400°.

The relation between the viscosity and pressure in the apparatus after this heating is not quite the same as before, showing that the gas liberated has not quite the composition of atmospheric air. It is therefore necessary to exhaust again to a very high point, and then to add a little air (some should always be retained in the air-traps for this purpose) and again exhaust. After this treatment it is practicable to attain much higher exhaustions.

Another very necessary precaution is to allow a sufficient time to elapse, after the pump has stopped, before the observations are taken. At low exhaustions the pressure in the apparatus equalises in a few minutes; but when the tension is reduced to a few millionths of an atmosphere, twenty minutes or half an hour must pass before the pressures in the McLEOD globe and the viscosity bulb are uniform. The normal pressure is always considered to be 760 millims. When the apparatus is full of gas the interior pressure is easily brought to 760 millims., whatever be the height of the barometer, by raising or lowering the mercury reservoir, and so varying the height of the mercury in the tube supplying the fall tubes.

652. A noteworthy point in connexion with the elasticity of glass is observed on the curves in diagram C. They are not continued beyond the 0·02 M exhaustion, but the general form of the curves indicates that, if they were produced beyond the limits of the observations, they would cut the line representing the absolute vacuum. The curves representing the repulsion accompanying radiation evidently go up to the zero point, showing that at an absolute vacuum there would be no repulsion. The curves of viscosity cannot, however, be supposed to end at the zero point

without a sudden change in direction. They evidently meet the top line of zero pressure long before the logarithmic decrement of 0·00 is reached. This means that in an absolute vacuum there would still be a measurable amount of viscosity. This is probably due to the viscosity of the glass torsion fibre, for it has been ascertained that glass is not perfectly elastic, but will take a permanent *set* if kept under constraint for a considerable time. I give an instance which has come under my own notice. In 1862 I purchased a piece of glass lace, and some spun glass from which the lace was made. The spun glass is in long straight threads, about 0·001 inch diameter, and has occasionally been used for torsion fibres. The fibres of which the lace was made were originally straight, but the twists and bends in which they have been kept for eighteen years have permanently altered their direction, and on dissecting a portion of the lace the component fibres remain distorted and bent, even when free to resume their original shape.

Were glass perfectly elastic the logarithmic decrement in an absolute vacuum would probably be equal to *zero*: there would then be no diminution in the arc of vibration, and the torsion fibre once set swinging would go on for ever.

VISCOSITY OF AIR.

653. The mean of a very large number of closely concordant results gives as the logarithmic decrement for air for the special apparatus employed, at a pressure of 760 millims. of mercury and a temperature of 15° C., the number 0·1124. According to MAXWELL the viscosity should remain constant until the rarefaction becomes so great that we are no longer at liberty to consider the mean free path of the molecules as practically insignificant in comparison with the dimensions of the vessel.

My observations show that this theoretical result of MAXWELL'S is sufficiently near the actual fact in air to confirm the accuracy of his reasoning, although there is a variation showing that disturbing influences are at work which make the coefficient of viscosity (taken as proportional to the logarithmic decrement) not quite constant.

The results are embodied in the following table and diagrams.

The first half of Table I. gives the viscosity of air, in so far as it is represented by the log. decs., at pressures intermediate between 760 millims. and 0·76 millim. (1000 millionths of an atmosphere). In order to avoid the inconvenience of frequent reference to small fractions of a millimetre, I now take the millionth of an atmosphere* (=M) as the unit instead of the millimetre. The second half of the table is therefore given in millionths, going up to an exhaustion of 0·02 millionth of an atmosphere, the highest point to which I have carried the measurements, although by no means the highest exhaustion of which the pump is capable.

At the high exhaustions, in addition to the observed results, I have given the calculated mean free path of the molecules.

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* 1 M=0·00076 millim.; 1315·789 M=1 millim.

654. TABLE I.—Log. dec.* of dry air at pressures between 760 millims. and 0·76 millim. Temp. 15° C.

Pressure. Millims.	Log. decrement.
760	0·1124
755	0·1123
750	0·1121
730	0·1117
702	0·1110
685	0·1105
664	0·1100
645	0·1096
627	0·1091
605	0·1086
587	0·1082
572	0·1078
550	0·1073
511	0·1065
495	0·1062
475	0·1057
440	0·1050
409	0·1043
395	0·1040
385	0·1038
350	0·1032
324	0·1027
301	0·1022
272	0·1019
254	0·1016
233	0·1013
210	0·1010
180	0·1008
155	0·1006
120	0·1004
112	0·1003
68	0·1002
47	0·1001
26	0·1001
12	0·1000
4	0·1000
3	0·0994
1	0·0988
0·76	0·0988

Pressure. Millims.	Time of one complete vibration. Seconds.
772	11·20
633	11·10
534	11·05
425	11·05
336	10·96
260	10·88
155	10·90
106	10·80
74	10·80
50	10·80
23	10·72
8	10·76
1	10·70
0·68	10·76
M.	
235·0	10·76
94·0	10·72
47·0	10·70
13·5	10·68
9·0	10·72
2·5	10·72
1·3	10·72
	&c.
	&c.

---

* In making the experiments described in this paper, and, indeed, in writing the greater part of the paper itself, I have assumed that the logarithmic decrement was a measure of the relative viscosities of the different gases, at least until exhaustions were reached at which the viscosity begins to break down. Professor STOKES has shown that this is not the case except under certain restrictions, and in a note appended to this paper he has deduced the results which follow from my experiments when reduced according to strict dynamical principles.—W. C., August 11, 1881.

TABLE I. (continued).—Log. dec. of dry air at pressures between 1000 M and 0·02 M. Temp. 15° C.

Pressure. M	Log. decrement.	Repulsive force of radiation.	Mean free path of molecules. Millims.
1000·0(=0·76 m.m.)	0·0988	2·4	0·10
905·0	0·0983	3·2	0·11
736·0	0·0975	3·5	0·14
590·0	0·0971	4·5	0·17
495·0	0·0966	5·5	0·20
385·0	0·0960	8·5	0·26
300·0	0·0952	10·0	0·33
248·0	0·0943	12·0	0·40
219·0	0·0937	14·0	0·46
183·0	0·0930	16·5	0·55
165·0	0·0926	18·0	0·61
157·0	0·0925	19·5	0·64
135·0	0·0907	21·3	0·74
116·0	0·0892	24·5	0·86
100·0	0·0876	27·0	1·0
93·0	0·0866	29·0	1·1
81·0	0·0842	31·1	1·2
79·0	0·0840	31·5	1·3
72·0	0·0824	32·9	1·4
68·0	0·0817	33·5	1·5
62·0	0·0799	34·6	1·6
53·0	0·0774	37·0	1·9
39·0	0·0710	41·4	2·6
36·0	0·0695	42·5	2·8
29·0	0·0657	42·6	3·4
24·0	0·0620	41·2	4·2
19·0	0·0577	38·8	5·3
13·0	0·0500	30·9	7·7
11·0	0·0460	27·1	9·1
8·0	0·0390	22·3	12·5
7·2	0·0372	20·2	13·9
5·9	0·0337	17·0	16·9
4·1	0·0281	13·1	24·4
3·4	0·0256	11·5	29·4
2·6	0·0225	8·5	38·4
1·9	0·0198	7·1	52·6
1·3	0·0175	4·2	76·9
1·0	0·0161	2·1	100·0
0·55	0·0144	2·0	181·8
0·46	0·0135	1·7	217·4
0·22	0·0118	1·4	454·5
0·14	0·0114	1·0	715·9
0·06	0·0097	0·7	1666·6
0·02	0·0072	0·5	5000·0
	0·00537	?	

655. The pressure and the logarithmic decrement columns are obtained in the following way:—Observations are taken first at frequent intervals during exhaustion, sufficient time being left between the successive experiments to allow the slight heating due to internal friction, and the cooling due to rarefaction to become equalised. Air is then slowly admitted through the drying tubes, and another series is taken. When several hundreds of such determinations are accumulated, they are collected into groups, from which the mean pressure and mean logarithmic decrement for the group have been taken.

In this table I have also given the measurements of the repulsion exerted on the blackened end of the mica plate by a candle-flame placed 500 millims. off. The repulsion due to radiation commences just at about the same degree of exhaustion where the viscosity begins to decline rapidly, and it principally comes in at the exhaustion above 1000 M. The observations are thus conducted:—The torsion fibre and plate being at rest, the shutter obscuring the candle is suddenly removed, and the light allowed to shine on the blackened mica for a series of seven or eight oscillations. The radiant force deflecting the mica (not including the torsion) is by no means constant, but mounts up during a very appreciable time. The motion of the mica is one of swinging about a point which itself changes more or less slowly with the time. This force begins from nothing, rapidly increases, and attains nearly its full value in a second or two after the candle is let on. The force depends on the heating of the surface of the swinging body by the previous action of the candle.

The path of the index ray is of the following character:—

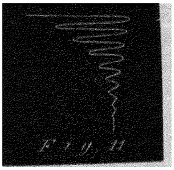
Fig. 11

Starting from zero, the mechanical force drives the index to division 88; thence it swings back to 29; the next swing brings it to 83, and so on. The problem is, from the varying arcs of oscillation, to find the true zero, or the point at which the index ray would rest supposing the force of repulsion to be constant and inertia absent.

656. Professor Stokes, to whom I am greatly indebted for assistance and advice on the mathematics of this subject, has provided a method for calculating the zero for the swing produced by the candle-flame. Register the stopping-points when the candle is let on in successive swings; and in an adjacent experiment find the logarithmic decrements, without the candle.

Let l be the logarithmic decrement, N the number whose logarithm is l. Divide the successive arcs by $1+N$, and apply the results to the readings of the second ends of the arcs, to get the equilibrium readings for the force arising from the candle supposed constant during that swing, and equal to what it was in the middle of the swing.

The following is an example of the observations which I have plotted above, treated in this way:—

Log. dec. 0·066; L. $\overline{1+N}$ 0·335.

Reading. I.	Arc. II.	Log. arc. III.	Less 0·335. IV.	Number. V.	Zero for swing. VI.
88	88	1·944	1·609	40·6	47·4
29	59	1·771	1·436	27·3	56·3
83	54	1·732	1·397	24·8	58·2
32	51	1·708	1·373	23·6	55·6
75	43	1·633	1·298	19·9	55·1
40	35	1·544	1·209	16·2	56·2
68	28	1·447	1·112	12·9	55·1
				Mean	55·9

The numbers in column VI. are obtained by applying column V. to column I., by subtraction and addition alternately. The first figures are marked off with a line, because the method gives only a rude approximation for the first swing, as the whole of the force, very nearly, has grown during this swing.

657. With a high logarithmic decrement the process just given should always be used; but with a low logarithmic decrement, as in the example, it may be simplified by taking the mean of consecutive arcs, then the half means, and then applying these + and − to the successive readings, as in the following example:—

Reading. I.	Swing. II.	Arithmetical mean of II. III.	Half of III. IV.	Applied + and − to I. = approximate calculated position of equilibrium. V.
0				
	88			
88		73·5	36·7	52·1
	59			
29		56·5	28·2	57·2
	54			
83		52·5	26·2	56·8
	51			
32		47·0	23·5	55·5
	43			
75		39·0	19·5	55·5
	35			
40		31·5	15·7	55·7
	28			
			Mean	56·1

I have cut off the first swing, which is clearly out of series. The slight diminution of the last three, as compared with the second and third numbers, may not improbably be real, though the slight uncertainty as to the second makes it doubtful. If it is real, it is probably due to the heating of the hinder face of the mica by conduction from the front face; so that the permanent difference of temperature is a little less than was the difference at a somewhat earlier stage.

658. There is great satisfaction to the mind when it is possible to plot a long series of figures as a curve on paper. Indeed, this is almost the only way in which the meaning of the results can be fully realised. It has been found impracticable to place the curves together in one table, inasmuch as the diagrams necessary to enable the results to be understood, and without which columns of figures are almost valueless, have to be on a scale of increasing magnification. On diagram A (Plate 56) are shown the results recorded in the first part of the table down to a pressure of 0·76 millim. plotted as a curve (marked "air"), the ordinates being the logarithmic decrements, and the abscissæ the pressures in millims. of mercury. The total barometric height, or one atmosphere, is represented by a length of 152 millims.: each millim. on the scale therefore represents 5 millims. pressure of mercury. Gaps in the series have not been filled up by interpolation.

Starting from the logarithmic decrement 0·1124 at 760 millims., it diminishes very regularly, but at a somewhat decreasing rate. Between 50 millims. and 3 millims. the direction is almost vertical, and it will be noticed that a great change in the uniformity of the viscosity curve commences at a pressure of about 3 millims. (shown in the upper part of the curve on diagram A), where the direction suddenly changes and goes off almost horizontally. At this point the previous approximation to MAXWELL's law begins to fail, and further pumping considerably reduces the logarithmic decrement.

659. To render the gradation of the air curve more uniform, it is now necessary to considerably magnify the scale. Diagram A is on so small a scale that 152 millims. represent the whole barometric column, but the scale to which diagram B is drawn is that of 1 millim. to 5 millionths of an atmosphere, so that the whole atmospheric pressure would be represented by a vertical height of 200 metres. One-thousandth part of this, representing the top 200 millims. of the scale, is all that I give in the compass of this diagram. At the lower part of diagram B (Plate 57) is given a highly condensed scale of the viscosity curve between 760 millims. and 0·76 millim. To bring this condensed scale to the same proportion as that of the rest of the diagram it would have to be 200 metres instead of only 10 millims. long.

Starting from 1000 M the diminution of viscosity is very slight until the exhaustion reaches about 250 M; after that the viscosity gets less with increasing rapidity, and falls away quickly after 35 M is reached.

660. It will be seen, from the almost horizontal character of the upper part of the curve, that the scale is not even now large enough to bring it out properly. I have therefore in diagram C (Plate 58) again lengthened the scale, so that one atmosphere is represented by 5000 metres, and of this I have taken the top 300 millims., giving

the curve for the portion between 60 M and 0·02 M. On this scale one millionth of an atmosphere is represented by 5 millims. vertical height.*

The "air" curve on diagram C shows the further diminution in the viscosity of air. At the foot of the diagram a straight diagonal line starts from the logarithmic decrement 0·1124 at 1,000,000 M (one atmosphere) up to the logarithmic decrement 0·098 at 1000 M, and a second line to the logarithmic decrement 0 0794 at 60 M. The slope of these lines is that of the condensed curves in diagrams A and B, and is greatly exaggerated, since in order to get the true slope proper to join on to the rest of the curve the lines should be respectively 200 metres and 5000 metres long instead of 30 millims.

661. The dotted curves on the left of diagrams B and C show the variation in the force of repulsion under the influence of radiation; the force gradually increases until it reaches a maximum between 25 and 40 M, when it rapidly sinks. This curve agrees very well with similar curves given in former papers, representing the action of light on the radiometer† (334, 382, 481).

The close agreement between the loss of viscosity and the increased action of radiation is very striking up to the 35 millionth, when the repulsion curve turns round and falls away as rapidly as the viscosity.

662. On the right side of diagrams B and C I have given a third curve. The abscissæ represent the mean free paths of the molecules at the various pressures used as ordinates. The mean free paths being very small, to make them comparable with the other dimensions adopted on diagram B, I have multiplied their lengths by 20. Thus at 1000 M the mean free path is 0·1 millim., and at 100 M it is 1·0 millim. In diagram B, drawn on a scale of 20 to 1, these become respectively 2·0 and 20 millims., but in diagram C, I have given the actual lengths of the free paths of the molecules.

The curves of increasing mean free path and diminishing viscosity closely agree. This agreement is more than a mere coincidence, and is likely to throw much light on the cause of viscosity of gases.

RESISTANCE OF AIR TO THE PASSAGE OF AN INDUCTION SPARK.

663. In describing the apparatus (fig. 1, par. 641) I have mentioned that it is furnished with a resistance tube, *l*. This tube is provided with aluminium poles, so that at any pressure the appearance presented by the induction spark when passed through the tube may be observed. Since the publication of my researches on the phenomena presented by the passage of the induction discharge through high vacua, the present results—which, although never published, precede by a year or two those

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* To give some idea of the high exhaustions at which accurate measurements can be taken, I may mention that the highest exhaustion on the table—0 02 M—bears the same proportion to the ordinary atmospheric pressure that 1 millim. does to 30 miles, or, converting it into time, that 1 second bears to 20 months.

† Phil. Trans., Part I., 1878: The Bakerian Lecture; Ditto, Part I., 1879. Proc. Roy. Soc., vol. xxv., p. 305.

just mentioned—have lost much of their interest. I will therefore only briefly mention some of the results at a few pressures.

The poles, formed of aluminium, are 4 millims. apart at their ends. The coil when in full work gives a 6-inch spark, but a small battery was used in order to reduce the full striking distance in air to 85 millims.

664. At an exhaustion of 295 M the resistance of 4 millims. of rarefied air in the tube between the poles is equal to 2 millims. outside; that is to say, when the outer terminals of the coil are 2 millims. apart it is a matter of indifference whether the spark passes between them or through the vacuum tube. There is much reddish-violet light round the poles, the appearance being that of ordinary vacuum poles. Traces of green phosphorescence flash out occasionally.

At an exhaustion of 82 M the violet light round the poles has disappeared. The + end of the tube is filled with hazy violet light, and there is a small spark and a brush at the tip of the — pole. Faint green phosphorescence is seen on the glass round the — pole. The resistance of the vacuum is now equal to 24 millims. air space.

Exhaustion 27 M. The violet diffused light is much fainter, and the green phosphorescence is getting stronger. Resistance of the vacuum = 24 millims. of air.

Exhaustion 4 M. No violet light is to be seen. The green light is very strong, and a diffused greenish-yellow light fills the + end of the tube. Resistance of the vacuum = 53 millims. of air.

Exhaustion 1 M. The green phosphorescence is like the last, but somewhat stronger. The spark and brush at the tip of the — pole, first noticed at 82 M, are still there. Resistance of the vacuum = 85 millims. of air space.

Exhaustion 0·5 M. The appearance is similar to the last, but the spark and brush have gone from the — pole. The wires connecting the coil with the tube were now carefully insulated. When the poles are kept apart the striking distance in air (85 millims.), the spark often strikes across rather than pass through the tube, but on separating the poles beyond this distance the discharge flashes intermittently through the tube, sometimes keeping up for one or two seconds.

When thus passing through the tube the flashes of green phosphorescence are very brilliant, particularly at the + end. During the intervals between the flashes a faint band of green light is seen round the inner surface of the tube at the tip of the — pole.

Exhaustion 0·02 M. With the battery and coil as hitherto no discharge whatever passes. On increasing the battery power till the striking distance in air was 100 millims. the spark occasionally passed through as an intermittent flash, bringing out faint green phosphorescence on the glass round the end of the — pole.

665. The terminals of the tube and wires leading to them must be well insulated, and kept away from any part of the apparatus. If this precaution is neglected a spark breaks through from one of the wires to some part of pump or connecting tubes,

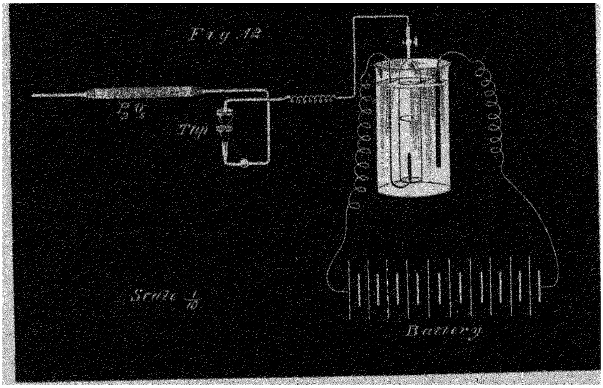
and perforates the glass. The perforation is sometimes so fine as to cause the greatest trouble in finding the leakage, which only shows itself by the inability of the pump to give a good vacuum. Left to itself the logarithmic decrement slowly rises, the repulsive force of the candle increases to its maximum, and then slowly diminishes to zero, the logarithmic decrement continuing to rise till ultimately the internal and external pressures are identical. With a fine perforation it takes several days to go through these phases, and they occur with such slowness and regularity as to afford opportunities for valuable observations.

On one occasion I obtained a much higher exhaustion than 0·02 M. I could not measure it, but from the repulsion by radiation and the low logarithmic decrement (0·00537) I should estimate it at about 0·01 M. The terminals of the vacuum tube and wires leading to them were well insulated, and the full power of a coil giving a 20-inch spark was put on to it. At first nothing was to be seen. Then a brilliant green light flashed through the tube, getting more and more frequent. Suddenly a spark passed from a wire to the glass tube, and broke it, terminating the experiment.

Since these experiments I have frequently got vacua as high, and even higher, but I have never seen one that would long resist the 20-inch spark from my large coil.

VISCOSITY OF OXYGEN.

666. The series of experiments with air show a complete history of its behaviour between very wide limits of pressure. It became interesting to see how the two components of air, oxygen and nitrogen, would behave under similar circumstances.



Experiments were therefore instituted exactly as in the case of dry air, but with the apparatus filled with pure oxygen.

For these experiments the oxygen gas is prepared by the electrolytic decomposition of water containing sulphate of copper dissolved in it, a copper plate being used for the negative pole and a platinum plate for the positive pole. The apparatus is shown in fig. 12. The reduction of the copper effectually obviates any chance of hydrogen coming off and contaminating the oxygen. The gas is received direct into a bell receiver in the acid liquid, and is thence transferred to the apparatus through a vacuum tap beyond the drying tubes, *w*, Plate 55, fig. 1. The oxygen is always kept under a considerable pressure of water, and is only prepared shortly before it is wanted. The gas is passed into the apparatus very slowly through a considerable length of phosphoric anhydride.

The following table gives the numerical results obtained :—

667. TABLE II.—Log. dec. of oxygen gas at pressures between 760 millims. and 0·76 millim. Temp. 15° C.

Pressure. Millims.	Log. decrement.	Time of one complete vibration.	
		Millims.	Seconds
760	0·1257	775	11·4
720	0·1247	753	11·3
668	0·1232	623	11·2
632	0·1222	543	11·1
564	0·1205	443	11·0
562	0·1204	343	10·9
496	0·1168	202	10·9
458	0·1178	158	10·9
340	0·1152	93	10·8
318	0·1148	63	10·8
309	0·1146	27	10·8
301	0·1145	7	10·76
249	0·1136	1	10·76
200	0·1130		&c.
159	0·1126		&c.
150	0·1125		
120	0·1123		
103	0·1122		
80	0·1120		
72	0·1120		
61	0·1119		
47	0·1119		
38	0·1118		
30	0·1118		
26	0·1119		
19	0·1119		
16	0·1121		
12	0·1122		
7·5	0·1124		
3·8	0·1121		
1·5	0·1115		
1·4	0·1111		
1·1	0·1115		
1·0	0·1115		
0·85	0·1107		
0·76	0·1102		

TABLE II. (continued).—Log. dec. of oxygen gas at pressures between 1000 M and 0·3 M. Temp. 15° C.

Pressure. M	Log. decrement.	Repulsive force of radiation.
1000·0 (=0·76 m.m.)	0·1102	12·
803·0	0·1093	12·
658·0	0·1088	13·
623·0	0·1086	13·
613·0	0·1085	13·
360·0	0·1070	13·
297·0	0·1058	14·
190·0	0·1038	20·
171·0	0·1033	21·
110·0	0·0988	31·
82·0	0·0940	35·
70·0	0·0912	38·
48·0	0·0840	45·
31·0	0·0744	44·
28·0	0·0724	44·
22·0	0·0670	40
16·0	0·0621	35·
12·0	0·0585	30·
4·0	0·0433	14·
1·6	0·0348	7·
0·3	0·0302	2·

668. These figures, plotted as curves on diagrams A, B and C, show a great similarity to the air curve. Like it, the viscosity sinks somewhat rapidly between pressures from 760 millims. to about 75 millims. It then remains almost steady, not varying much till a pressure of 16 millims. is reached. Here, however, it turns in the opposite direction, and steadily increases up to 1·5 millim. It then diminishes again, and at higher exhaustions it rapidly sinks. This increase of viscosity at pressures of a few millimetres has been observed in other gases, but only to so small an extent as to be scarcely beyond the limits of experimental error. In the case of oxygen, however, the increase is too great to be entirely attributable to this cause.

669. Oxygen has more viscosity than any gas I have yet examined. The viscosity of air at 760 millims. being 0·1124, the proportion between that of air and oxygen, according to these results, is 1·1185. GRAHAM[*] makes it 1·1099 (706). I have been unable to find viscosity results of oxygen given by any other observers.

This proportion of 1·1185 holds good (allowing for experimental errors) up to a pressure of about 20 millims. Between that point and 1 millim. variations occur, which I have not been able to trace to any assignable cause: they seem large to be put down to "experimental errors." The discrepancies disappear again at an exhaustion of about 1 millim., and from that point to the highest hitherto reached the proportion of 1·1185 is fairly well maintained.

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[*] *Loc. cit.*, p. 179.

670. The curves plotted from the observations of the repulsive force of radiation are given in dots at the left side of diagrams B and C. Repulsion commences at a somewhat lower exhaustion, and remains much higher than in air until an exhaustion of 22 **M** is reached, when the two curves become practically identical.

VISCOSITY OF NITROGEN.

671. Nitrogen is prepared by heating a solution of nitrite of ammonia. If proper precautions are taken to avoid intermixture of air, this process yields it in a very pure state. It is collected in the bell receiver shown in fig. 12 (666). To eliminate aqueous vapour the gas is passed through phosphoric tubes, and then through the tap into the viscosity apparatus, as explained in the case of oxygen. The last traces of aqueous vapour are removed from the nitrogen by allowing it to pass in a very slow stream through a series of tightly-packed phosphoric anhydride tubes, on its way to the viscosity bulb.

Nitrogen was also prepared by burning phosphorus in a confined space of air, and also by the copper turnings and ammonia method; but neither of these processes yielded a gas which gave uniformly such concordant results as did the nitrite of ammonia process.

The following are the tabulated results of observations with nitrogen :—

672. TABLE III.—Log. dec. of nitrogen gas at pressures between 760 millims. and 2·8 M. Temp. 15° C.

Pressure. Millims.	Log. decrement.
760	0·1092
717	0·1081
692	0·1074
624	0·1059
582	0·1048
542	0·1038
474	0·1023
403	0·1010
324	0·0996
282	0·0990
245	0·0985
218	0·0980
172	0·0976
165	0·0974
143	0·0974
122	0·0973
117	0·0970
105	0·0973
100	0·0971
87	0·0971
79	0·0971
60	0·0972
50	0·0970
44	0·0969
29	0·0970
21	0·0970
14	0·0969
8	0·0969
6	0·0970
2·1	0·0961
1·4	0·0962
0·85	0·0962
0·76	0·0960

TABLE III. (continued).—Log. dec. of nitrogen gas at pressures between 1000 M and 2·8 M. Temp. 15° C.

Pressure. M	Log. decrement.	Repulsive force of radiation.
1000·0 (=0·76 m.m.)	0·0960	1·
610·0	0·0941	2·
459·0	0·0934	4·
345·0	0·0930	3·
188·0	0·0894	8·
125·0	0·0867	15·
84·0	0·0820	23·
58·0	0·0770	28·
47·0	0·0730	30·
26·0	0·0600	25·
13·0	0·0420	17·
9·6	0·0351	14·
8·3	0·0318	13·
5·8	0·0257	9·
3·3	0·0207	3·
2·8	0·0178	1·

673. The proportion between the viscosities of nitrogen and air at a pressure of 760 millims. is according to these experiments 0·9715. GRAHAM made it 0·971 (706).

674. A comparison of the air curves with those given by oxygen and nitrogen gives some interesting results. The composition of the atmosphere is, by bulk,

Oxygen	20·8
Nitrogen	79·2
	100·0

The viscosity of the mixed gases is almost exactly that which would be inferred from the composition: thus at 760 millims.—

$$\frac{20{\cdot}8 \textit{ vis. } \mathrm{O} + 79{\cdot}2 \textit{ vis. } \mathrm{N}}{100} = \textit{vis. } \text{air},$$

$$\frac{20{\cdot}8\,(0{\cdot}1257) + 79{\cdot}2\,(0{\cdot}1092)}{100} = \quad ,,$$

$$= \frac{2{\cdot}61456 + 8{\cdot}64072}{100} = 0{\cdot}11255,$$

a result closely coinciding with 0·1124, the experimental result for air. Up to an exhaustion of about 30 M the same proportion between the viscosities of air, oxygen, and nitrogen is preserved with but little variation. From that point great divergence occurs between the individual curves of the three gases. The difficulties of obtaining concordant observations at these high vacua are very great, and the unavoidable errors of experiment are, I think, sufficient to account for any divergence between theory and observation.

OBSERVATIONS ON THE SPECTRUM OF NITROGEN.

675. Spectrum observations during exhaustion give the following results:—

At 55 millims. pressure the band spectrum of nitrogen commences to be visible. The red and yellow bands are easily seen, and the green and blue are exceedingly faint. As the pressure grows less the bands become more distinct, until at 1·14 millim. the band spectrum is at its brightest.

At a little higher exhaustion a change comes over the spectrum, and traces of the line spectrum are observed.

At 812 M both the band and the line spectrum can be seen very brilliantly.

At 450 M the line spectrum is seen in great purity. As the exhaustion becomes higher the lines commence to disappear at the two ends of the spectrum.

At 188 M the lines below λ 610 at the red end, and above λ 400, cease to be visible.

At 94 M a bright greenish-yellow line is visible at about λ 567.

At 55 M this greenish-yellow line is still very prominent. The red lines have disappeared altogether, and the highest blue line visible is one at λ 419. The line λ 567 varies much in visibility; sometimes it cannot be seen, whilst at others it is very visible. Thus—

At 40 M the line λ 567 has quite disappeared.

At 17 M line λ 567 is visible again, being the most prominent line left.

At 12 M line λ 567 is not seen, although several other green and blue lines are left.

At 3 M only three lines are visible in the green, and these are very faint.

At 2·8 M line λ 567 is detected again.

At 2 M only traces of one or two lines can be seen, the faint light of the lines being overpowered by the green phosphorescence of the glass.

Line λ 567 has been seen on several occasions at high exhaustions when the gas under examination has been mixed with a little air. It is probably a nitrogen line, for one of the most brilliant nitrogen lines has a wave length of λ 567·8 (THALÉN), λ 568·0 (HUGGINS), or λ 568·1 (PLUCKER), and my interpolation curve is not sufficiently accurate to enable me to say that the line I have entered in my notes as being at λ 567 may not in reality be a trifle higher. The reason of its being only sometimes visible may be accounted for by a difference in the sensitiveness of the eye at different times, or by a difference in battery power. This, however, cannot be the whole explanation, for other lines are not found to vary in the same manner.

676. The curve of repulsion exerted by radiation is plotted in dots on diagrams B and C. It is much lower than in oxygen or air, and sinks rapidly after the maximum is passed.

VISCOSITY OF CARBONIC ANHYDRIDE.

677. Several processes have been tried for the preparation of this gas, to avoid the collection over water, and consequent risk of admixture with air. The best methods are the ignition of bicarbonate of soda, and the decomposition of marble by hydrochloric acid. The bicarbonate of soda plan has two great advantages: the generation of gas can be conducted in the sealed-up and exhausted apparatus, and nothing else but carbonic anhydride and water are given off. The amount of water, however, proves a serious inconvenience, as it uses up so much phosphoric anhydride,—therefore the production of the gas from marble and hydrochloric acid was finally employed.

The apparatus is shown in fig. 13.

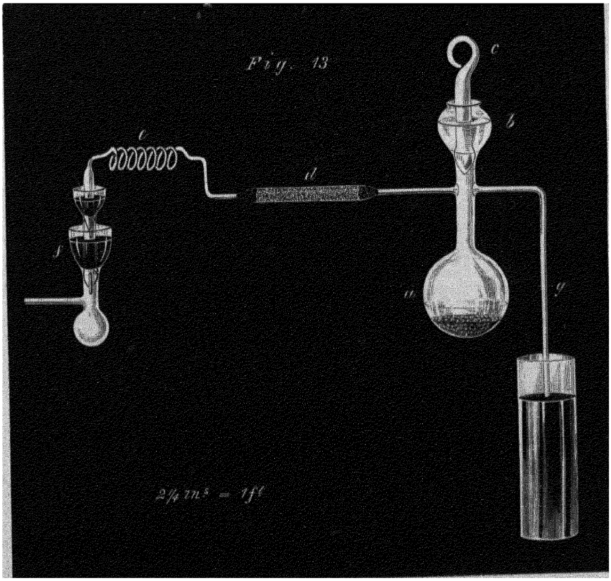

Fig. 13

The marble is contained in the glass flask, *a*, to the upper part of which a thistle-shaped funnel, *b*, with a ground stopper, *c*, is attached. The bulb, *b*, is filled with hydrochloric acid, and by loosening the stopper some of the acid is allowed to run down on the marble. The carbonic anhydride passes along the tube, *d*, which is tightly packed with powdered bicarbonate of soda to stop hydrochloric acid, through the flexible spiral, *e*, to the tap, *f*. On the other side of the flask the tube, *g*, dips into a tall glass containing mercury. The tap is connected with the viscosity apparatus, as shown in the general view in fig. 1.

Before introducing carbonic anhydride into the apparatus the gas is freely generated, and allowed to bubble up for some time through the mercury in the upper reservoir of the tap, the ground tube in it being loosened for this purpose. When it is supposed that all air is driven out of the flask and tubes by the carbonic anhydride, the tube is tightly pressed into the tap so as to close it, and the gas then escapes through the tube, *g*, under considerable pressure of mercury. The viscosity apparatus on the other side of the tap being already well exhausted, the tap is slightly turned to allow a very slow stream of carbonic anhydride to pass into the apparatus, generation of gas being kept up by feeding in hydrochloric acid through the stopper, *c b*. When the apparatus is filled with carbonic anhydride the tap, *f*, is turned off, and exhaustion is proceeded with. These operations are repeated at least three times, several hours being allowed to elapse after filling the apparatus before pumping out the gas, so as to let it soak well into all parts, and displace any other gas which may cling to the surfaces. After the last filling, observations are commenced. Several series are taken, and if, as is sometimes the case, the earlier ones are not uniform with the others, they are rejected.

In the following table I give the results of the observations with this gas:—

678. TABLE IV.—Log. dec. of carbonic anhydride at pressures between 760 millims. and 7·6 M. Temp. 15° C.

Pressure. Millims.	Log. decrement.
760	0·1035
750	0·1033
689	0·1022
665	0·1016
645	0·1012
640	0·1009
620	0·1006
605	0·1001
545	0·0982
540	0·0979
512	0·0970
501	0·0967
460	0·0948
425	0·0937
410	0·0931
403	0·0928
368	0·0918
363	0·0914
355	0·0911
332	0·0903
290	0·0890
282	0·0884
255	0·0875
240	0·0872
205	0·0858
180	0·0850
132	0·0835
124	0·0831
92	0·0826
71	0·0823
55	0·0822
50	0·0822
28	0·0821
11	0·0822
4·5	0·0819
2·3	0·0817
1·1	0·0816
0·76	0·0809

Pressure. Millims.	Time of one complete vibration. Seconds.
732	11·48
592	11·32
512	11·28
402	11·16
332	11·06
222	11·02
127	10·92
87	10·84
37	10·80
32	10·76
7	10·76
3	10·76
2	10·76
1	10·76
	&c.
	&c.

TABLE IV. (continued).—Log. dec. of carbonic anhydride at pressures between 1000 M and 7·6 M. Temp. 15° C.

Pressure. M	Log. decrement.	Repulsive force of radiation.
1000 (=0·76 m.m.)	0·0809	1·
588	0·0790	2·
523	0·0785	2·
389	0·0780	2·
345	0·0776	2·
228	0·0770	3·
177	0·0758	5·
158	0·0750	7·
91	0·0717	13·
66	0·0697	15·
58	0·0684	19·
40	0·0630	25·
32	0·0600	25·
15	0·0424	16·
10	0·0347	11·
9	0·0325	10·
7·6	0·0298	8·

679. The curves given in diagrams A, B, and C are plotted from the above observations. At first the curve seems to follow the same direction as the air curve. But at a pressure of about 620 millims. it slopes more rapidly till the pressure is reduced to about 50 millims., when the curve again takes the direction of the air curve. The total diminution between 760 millims. and 1 millim. is nearly double that of air.

The proportion between the viscosity of carbonic anhydride and air at 760 millims. is 0·9208. GRAHAM* found it to be 0·807, and KUNDT and WARBURG† 0·806. MAXWELL‡ made it 0·859 (706).

680. The curve showing the variation in the repulsive force of radiation, shown in dots in diagrams B and C, commences late and falls off early, the maximum being very inferior to that obtained in the other gases examined.

OBSERVATIONS ON THE SPECTRUM OF CARBONIC ANHYDRIDE.

681. Observations have also been taken with the spectroscope during the exhaustion of carbonic anhydride. The maximum brilliancy of the spectrum occurs at an exhaustion of about 300 M. After that it gets fainter; at about 75 M the blue band (λ 409 to λ 408) disappears; as the exhaustion gets higher the other bands vanish until, at

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* 'Chemical and Physical Researches,' p. 179.

† Phil. Mag., July, 1875.

‡ Phil. Trans., 1866, Part I., p. 257.

a vacuum of about 40 M, nothing is visible but the two lines λ 519 and λ 560. At higher exhaustions these lines disappear, and the phenomena of "radiant matter" commence.

VISCOSITY OF CARBONIC OXIDE.

682. This gas was prepared by heating oxalate of ammonia and strong sulphuric acid, and collecting the mixed carbonic oxide and carbonic anhydride over solution of caustic potash in the bell-jar shown in fig. 12 (666). Previous to entering the apparatus the gases are slowly passed through long tubes containing powdered caustic potash.

Carbonic oxide was also prepared by heating ferrocyanide of potassium with strong sulphuric acid, and passing the resulting gas through potash tubes as before. This is the preferable way to prepare the gas, for it is difficult to remove the last traces of carbonic anhydride when the oxalic acid process is used.

The gas is passed into the well-exhausted apparatus through the tap in the manner already described, and the exhaustion and filling are repeated a sufficient number of times. All usual precautions are taken, as described in the case of gases previously examined.

The following table gives the experimental results obtained :—

683. TABLE V.—Log. dec. of carbonic oxide, between pressures of 760 millims. and 0·76 millim. Temp. 15° C.

Pressure. Millims.	Log. decrement.
760	0·1092
687	0·1073
517	0·1031
426	0·1012
353	0·1000
266	0·0983
219	0·0979
125	0·0972
89	0·0971
79	0·0970
32	0·0971
18	0·0968
12	0·0968
10	0·0967
2·6	0·0969
1·4	0·0957
1·2	0·0958
0·76	0·0947

TABLE V. (continued).—Log. dec. of carbonic oxide between pressures of 1000 M and 6·5 M. Temp. 15° C.

Pressure. M	Log. decrement.	Repulsive force of radiation
1000 (=0·76 m.m.)	0·0947	2·
829	0·0945	3·
629	0·0936	4·
474	0·0925	5·
397	0·0921	6·
200	0·0910	13·
188	0·0907	14·
126	0·0877	18·
86	0·0838	20·
55	0·0778	21·
42	0·0716	22·
38	0·0692	22·
31	0·0653	21·
22	0·0580	18·
13	0·0474	14·
12	0·0448	13·
6·5	0·0305	7

684. These numbers are plotted as curves in the diagrams A, B, and C. They are remarkable as showing an almost complete identity with those of nitrogen on the same diagrams, both in position and shape. The viscosity at 760 millims. is in each case 0·1092. According to GRAHAM the proportion between the viscosities of air and carbonic oxide is 0·971, nitrogen giving the same (706). With my apparatus the proportion is 0·9715.

Like that of nitrogen, the curve of carbonic oxide is seen to be vertical,—*i.e.*, assuming the curve to represent the viscosity, the gas obeys MAXWELL's law,—at pressures between 90 millims. and 3 millims. The vertical portion in nitrogen is at a little higher pressure: between 100 millims. and 6 millims.

685. The curve of repulsion resulting from radiation, shown in dots in diagrams B and C, is lower in carbonic oxide than in any other gas examined, and, unlike the other gases, there is no sudden rise to a maximum at about 40 M. At lower exhaustions the curve is, however, higher than it is in nitrogen.

OBSERVATIONS ON THE SPECTRUM OF CARBONIC OXIDE.

686. During exhaustion observations were continued on the variations in the spectrum. The ordinary band spectrum is first seen with a few sharp lines terminating the bands.

At 12 millims. pressure a sharp green line is first seen, λ 515. This line rapidly grows brighter as exhaustion continues, and then fades out; it is last seen at a pressure of about 0·9 millim. This line is probably the bright oxygen line, the wave-length of which is given by PLUCKER at λ 514·4.

At a pressure of 2·8 millims. the spectrum agrees in appearance with the "Carbon No. 2" in WATTS's 'Index of Spectra.'

At 553 M the bands between the sharp lines appear to be breaking up into masses of fine lines.

At 211 M these fine lines are distinctly visible. The brightness of this spectrum is now near its maximum.

At 100 M the general spectrum is growing faint, but a sharp green line at λ 534 makes its appearance by fits and starts. This is coincident with PLUCKER's bright oxygen line λ 534.

After this degree of exhaustion the spectrum rapidly gets fainter. The line λ 534 soon disappears, and the carbon lines also go one after the other, until at an exhaustion of 4 M only two lines are visible, λ 560 and λ 519.

VISCOSITY OF HYDROGEN.

687. Hydrogen gas is prepared in the apparatus shown in fig. 12 (p. 408) by the electrolytic decomposition of well-boiled water acidulated with pure sulphuric acid, a plate of amalgamated zinc being used for the positive pole, and platinum for the negative pole. The gas is collected in a bell receiver and passed into the apparatus when required, exactly as described under *Oxygen* (666). Hydrogen is also introduced into the apparatus in another way. In the tube, I (Plate 55, fig. 1), are pieces of palladium-foil electrolytically saturated with hydrogen; the foil retains the hydrogen even in a vacuum, and when desired gives it out on being heated with a lamp.

Before commencing experiments with hydrogen the apparatus must be exhausted up to the highest point, and then very slowly filled with the gas. Exhaustion is again carried out to a high point, and the slow filling with hydrogen repeated. It is then advisable to allow all to remain undisturbed for at least twenty-four hours, to permit the hydrogen to soak into the apparatus and displace any other gas which may be condensed on the surface of the glass. The whole is again exhausted, and then a third time filled with hydrogen.

Preliminary observations are now taken for viscosity; and the logarithmic decrement at standard pressure and temperature, noted. The pumping out and refilling with hydrogen should now be repeated, observing the logarithmic decrement each time, until it is found that it no longer decreases in value; when this is the case accurate observations are commenced.

688. It has been found that hydrogen has much less viscosity than any other gas; the fact of the logarithmic decrement not decreasing by additional attempts at purification is the test of its being free from admixture. This method of ascertaining the purity of the gas, by the uniformity of its viscosity coefficient at 760 millims., is more accurate than collecting samples and analysing them eudiometrically (706).

Several series of observations in hydrogen have been taken. For a long time it was considered that hydrogen, like other gases, showed the same slight departure from MAXWELL's law of viscosity being independent of density; for the logarithmic decrement persistently diminished as the exhaustion increased, even at such moderate pressures as could be measured by the barometer gauge. Had it not been that the rate of decrease was not uniform in the different series of observations, I should have considered that this variation from MAXWELL's law was due to some inherent property of all gases. After working at the subject for more than a year, it was discovered that the discrepancy arose from a trace of water obstinately held by the hydrogen—an impurity which behaved as I explain farther on (699) in the case of air and water vapour. Since discovering this property, extra precautions (already described at the commencement of this paper (641)) have been taken to dry all gases before entering the apparatus.

The results are given in the following table :—

689. TABLE VI.—Log. dec. of hydrogen gas at pressures between 760 millims. and 0·76 millim. Temp. 15° C.

Pressure Millims.	Log. decrement	Pressure. Millims	Time of one complete vibration
760	0·0499	744	10·76
748	0·0501	614	10·76
582	0·0501	504	10·76
567	0·0498	314	10·76
484	0·0499	134	10·76
428	0·0500	64	10·76
423	0·0500	29	10·76
414	0·0499	8	10·76
399	0·0500	0·6	10·76
303	0·0500	7·9 M	10·76
301	0·0497		&c.
283	0·0498		&c.
268	0·0499		
212	0·0504		
209	0·0500		
201	0·0507		
193	0·0501		
174	0·0501		
148	0·0498		
128	0·0501		
108	0·0499		
103	0·0499		
101	0·0497		
96	0·0497		
76	0·0497		
72	0·0499		
61	0·0498		
33	0·0498		
22	0·0500		
17	0·0499		
14	0·0497		
11	0·0499		
9	0·0499		
7	0·0499		
6	0·0498		
2	0·0498		
1·8	0·0500		
1·5	0·0498		
1·0	0·0499		
0·76	0·0498		

TABLE VI. (continued).—Log. dec. of hydrogen gas at pressures between 1000 M and 0·16 M. Temp. 15° C.

Pressure. M.	Log decrement.	Repulsive force of radiation.	Mean free path of molecules.* Millims.
1000·0 (=0·76 m.m.)	0·0498	1·0	0·10
921·0	0·0498	1·0	0·11
526·0	0·0497	3·0	0·19
421·0	0·0496	4·0	0·24
330·0	0·0495	5·0	0·30
314·0	0·0493	8·0	0·32
234·0	0·0488	11·0	0·43
205·0	0·0486	14·0	0·49
179·0	0·0486	18·0	0·56
168·0	0·0485	19·0	0·59
155·0	0·0484	25·0	0·65
147·0	0·0482	28·0	0·68
135·0	0·0479	31·0	0·74
122·0	0·0475	37·0	0·82
110·0	0·0472	40·0	0·91
95·0	0·0466	44·0	1·1
79·0	0·0457	52·0	1·3
65·0	0·0446	60·0	1·5
59·0	0·0441	64·0	1·7
54·0	0·0435	66·0	1·8
48·0	0·0430	68·0	2·1
45·0	0·0422	69·0	2·2
41·0	0·0417	70·0	2·4
37·0	0·0408	69·0	2·7
33·0	0·0394	69·0	3·0
29·0	0·0384	67·0	3·4
26·5	0·0373	66·0	3·8
22·0	0·0358	60·0	4·5
20·0	0·0351	58·0	5·0
16·0	0·0333	52·0	6·3
14·5	0·0324	49·0	6·9
12·0	0·0304	45·0	8·0
8·0	0·0270	37·0	12·5
6·5	0·0253	31·0	15·4
5·0	0·0232	29·0	20·0
4·0	0·0214	26·0	25·0
2·6	0·0191	15·0	38·5
1·8	0·0172	10·0	55·5
1·5	0·0169	9·0	66·7
1·0	0·0157	7·0	100·0
0·37	0·0130	3·0	270·0
0·16	0·0118	2·0	625·0

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* For convenience I have taken the mean free path of the molecules of hydrogen the same as that of air, viz.: 0·0001 millim., at a pressure of 760 millims, and at 0° C. The actual figures, according to the most recent determinations, are—for air, 0·000095 millim.; and for hydrogen, 0·0000104 millim. The difference between these figures and 0·0001 is too small to show itself on the diagrams.

690. I have plotted the above observations in diagrams A, B, and C. The portion between 760 millims. and 0·32 millim. not varying in logarithmic decrement is represented by a vertical line, shown in diagrams B and C below the principal curve.

The remarkable character of hydrogen is its stiffness. It obey's MAXWELL's law almost absolutely up to an exhaustion of about 700 M. To this point the line of viscosity is almost perfectly vertical. It then commences to curve over, and when the mean free path begins to assume proportions comparable with the dimensions of the bulb, and approaches infinity, the viscosity curve in like manner draws near the zero line.

691. The repulsive force of radiation is higher in hydrogen than in any other gas. It commences at as low an exhaustion as 14 millims., but does not increase to any great extent till an exhaustion of 200 M is attained; it then rises rapidly to a maximum at between 40 and 60 M, after which it falls away to zero. The maximum repulsion exerted by radiation in hydrogen is to that in air as 70 to 42·6. This fact is now utilised in the construction of radiometers and similar instruments when great sensitiveness is required.

692. Taking the viscosity of air at 760 millims. as 0·1124, and hydrogen as 0·0499, the proportion between them is 0·4439. GRAHAM[*] gives the transpiration time for hydrogen as 0·4375 taking oxygen as unity, or 0·4855 taking air as unity (706).

693. Professor CLERK MAXWELL[†] found dry hydrogen to be much less viscous than air, the ratio of its viscosity to that of air being 0·5156. He says, referring to this result: "It appears, from the experiments of Mr. GRAHAM, that the ratio of the transpiration time of hydrogen to that of air is 0·4855, and that of carbonic acid to air 0·807. These numbers are both smaller than those of this paper. I think that the discrepancy arises from the gases being less pure in my experiments than in those of GRAHAM."

694. MM. KUNDT and WARBURG, in the paper already mentioned, assert that according to theory the viscosity should not begin appreciably to diminish as long as the layer of gas is thicker than fourteen times the mean length of path. They also declare that even when the thickness is as much as 300 times the mean length of path a perceptible diminution of the retarding forces commences, becoming greater as the pressure diminishes. They find the ratio between air and hydrogen to be 0·488, and the viscosity to vary according to the following table :—

Pressure in millims.	Log. decrement.
380·0	0·0652
20·0	0·0638
8·8	0·0629
2·4	0·0601
1·53	0·0557

Above this exhaustion they found the viscosity rapidly diminished, but no measurements of pressure were taken.

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[*] 'Chemical and Physical Researches,' by THOMAS GRAHAM, p. 179.

[†] "On the Viscosity or Internal Friction of Air and other Gases," Phil. Trans., 1866, Part I., p. 257.

695. The diminution of logarithmic decrement which MM. KUNDT and WARBURG obtained at moderate exhaustions, and their high ratio between the viscosity of hydrogen and air, is probably due to the presence of a trace of foreign gas, most likely water. They remark that: "On proceeding further to investigate the laws of gas-friction *below* the before-mentioned limit of rarefaction,—

$$\left(\frac{l}{d} > \frac{1}{14}\right),$$

we could not succeed, even with the most careful drying, in removing with sufficient completeness the last traces of aqueous vapour, which, insensible in the above experiments, with the *low* pressures here employed distorted the results. The presence of aqueous vapour was shown, *inter alia*, by this—that the damping moment for a vacuum (so we name a space filled with a gas of $\frac{1}{100}$th of a millim. pressure mixed with vapour) rose considerably when the apparatus was left to itself. This arose from water separating from the solid parts and evaporating into the vacuum. In consequence of this the theory cannot be quantitatively tested on the results obtained."

It is not unlikely that a trace of foreign gas or aqueous vapour was present in GRAHAM'S hydrogen, for he himself remarks that: "The small addition of 5 per cent. of air to hydrogen has a surprising effect in retarding the transpiration of that gas. The effect of 5 per cent. of air in retarding the rate of hydrogen is nearly four times greater than it should be by calculation. The experiment shows the effect which a small amount of impurity must have in deranging the transpiration rate of that gas."[*]

The same effect is noticed by MAXWELL, who says[†] that a small proportion of air mixed with hydrogen was found to produce a large increase of viscosity.

696. GRAHAM states that in his experiments the hydrogen "was prepared from zinc which contained no arsenic, and was passed through a wash-bottle containing oxide of lead dissolved in caustic soda, and dried by passing over asbestos moistened with oil of vitriol."[‡]

In another paper he speaks of drying the gases by means of chloride of calcium tubes, and pumice moistened with oil of vitriol. Guided by the light of recent experience I do not hesitate to affirm that by this means it is not possible to remove the last traces of aqueous vapour from hydrogen gas. It is not even enough to pass the gas through a tube containing phosphoric anhydride lying loose in it. The anhydride must be tightly packed in the tube, so as to offer considerable obstruction to the passage of the gas, which should pass through several feet of such desiccating tube (641, 688). Not until I adopted all these precautions to dry the gas was I able to get perfectly concordant results with different lots of hydrogen gas.

697. With each improvement in purification and drying I have obtained a lower value for hydrogen, and have consequently diminished the number expressing the

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[*] 'Chemical and Physical Researches,' by THOMAS GRAHAM, p. 123.

[†] "On the Viscosity or Internal Friction of Air and other Gases." Phil. Trans., 1866, Part I., p. 257.

[‡] 'Chemical and Physical Researches,' by THOMAS GRAHAM, p. 176.

ratio of the viscosity of hydrogen to that of air. In 1876 I found the ratio to be 0·508. In 1877 I reduced this ratio to 0·462. Last year, with improved apparatus, I obtained the ratio of 0·458, and I have now got it as low as 0·4439. This is much lower than GRAHAM's number 0·4855 deduced from transpiration. GRAHAM calls this number theoretical, but in discussing the subject it appears that he was quite prepared to find hydrogen depart from the theoretical law which seemed to govern some other of the gases. Thus at page 179 of the collected edition of his works, already quoted, GRAHAM writes: "The times of oxygen, nitrogen, carbonic oxide, and air are directly as their densities, or equal *weights* of these gases pass in equal times. Hydrogen passes in half the time of nitrogen, or twice as rapidly for equal volumes. . . . . The circumstance that the transpiration time of hydrogen is one-half of that of nitrogen indicates that the relations of transpirability are even more simple in their expression than the relations of density among gases."

OBSERVATIONS ON THE SPECTRUM OF HYDROGEN

698. In the description of the viscosity apparatus (641) I mentioned that a spectrum tube with capillary bore was attached to it at *k*. With this apparatus I have taken observations on the spectrum of hydrogen during the exhaustion, using a coil giving a 6-inch spark, and examining the spectrum with a dispersion sufficient to well separate the sodium lines.

No points of special interest were noticed. The red line ($\lambda=656·2$), the green line ($\lambda=486·1$), and the blue line ($\lambda=434$) were seen at their brightest at a pressure of about 3 millims., and after that exhaustion they begin to diminish in intensity. As exhaustion proceeds a variation in visibility of the three lines is observed. Thus at 36 millims. the red line is seen brightly, the green faintly, whilst the blue line cannot be detected. At 15 millims. the blue line is seen and the three keep visible till an exhaustion of 418 M is reached, when the blue line becomes difficult to see. At 38 M only the red and green lines are visible, the red being very faint. It is seen with increasing difficulty up to an exhaustion of 2 M, when it can be seen no longer. The green line now remains visible up to an exhaustion of 0·37 M, beyond which it has not been seen.

It is worthy of remark that, although when working with pure hydrogen the green line is always the last to go, it is not the first to appear when hydrogen is present as an impurity in other gases. Thus when working with carbonic anhydride insufficiently purified, the red hydrogen line is often seen, but I have never seen either the green or the blue line.

INFLUENCE OF AQUEOUS VAPOUR ON THE VISCOSITY OF AIR.

699. In the foregoing experiments many discrepancies were traced to the presence of moisture in the gas (688). The influence of aqueous vapour does not appear to be great when present in moderate amount in gas of normal density, but at high

exhaustions it introduces errors which interfere with the uniformity of the results. A series of experiments were accordingly undertaken to trace the special action of aqueous vapour when mixed with air.

The apparatus as represented in fig. 1 was slightly altered for this purpose, without, however, interfering with the parts affecting the measurements. The drying tubes of phosphoric anhydride were removed, and a small glass bulb filled with pure water was sealed on to the end of the arm *p* (the arm which, working between metal stops, gives motion to the viscosity bulb). The object of attaching the water bulb to this part of the apparatus was to allow the aqueous vapour readily to diffuse into the viscosity bulb, and drive the unsaturated air before it when the pump was worked.

The apparatus, thus arranged and full of air at the normal pressure, was left to itself for twenty-four hours, to allow aqueous vapour to diffuse through it. The temperature was uniformly 15° the whole time.

Observations were simultaneously taken with the candle flame, so as to measure the repulsion due to radiation in the presence of aqueous vapour.

The results are given in the following table :—

700. TABLE VII.—Log. dec. of moist air at pressures from 760 millims. to 0·1 millim, and lower. Temp. 15° C.

Pressure in millims.	Log. decrement.	Repulsion due to radiation.
760	0·1124	0·
700	0·1109	0·
600	0·1084	0·
500	0·1062	0·
400	0·1040	0·
300	0·1014	0·
200	0·0993	0·
100	0·0955	0·
75	0·0937	0·
50	0·0903	0·
40	0·0796	0·
20	0·0589	0·
16	0·0531	0·
15	0·0520	0·
13	0·0516	0·
11	0·0511	0·
8	0·0500	0·
7	0·0498	0·
5	0·0499	0·
3	0·0497	0·
1*	0·0497	0·
0·1	0·0484	0·
..	0·0441	1·
..	0·0432	1·
..	0·0419	4·
..	0·0406	5·
..	0·0390	9·

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* At about this degree of exhaustion the last drop of liquid water, which for some time had been rapidly evaporating, disappeared.

701. I was unable to take pressure after 1·0 millim., as the McLeod measuring apparatus does not give trustworthy indications when aqueous vapour is present. As soon as the pressure in the measuring tube rises above the tension of aqueous vapour, water condenses in it, and measurements can no longer be taken.

These results plotted as a diagram give the curve shown in diagram A, marked "Moist Air."

Up to a pressure of about 350 millims. the presence of aqueous vapour has little or no influence on the viscosity of air. The two curves are in fact superimposed. At this point, however, divergence commences, and the curve rapidly bends over, the log. dec. falling from 0·0903 to 0·0500 between 50 and 7 millims. pressure. Here it joins the hydrogen curve, and between 7 millims. and 1 millim. they appear to be identical.

702. These results are partly to be explained by the peculiar action of water vapour in the apparatus. At the normal pressure the amount of aqueous vapour present in the air, supposing it to be saturated, is only about 13 parts in a million, and the identity of the logarithmic decrement with that of dry air shows that this small quantity of water has no appreciable action on the viscosity. When the pump is set to work the air is gradually removed, whilst the aqueous vapour is kept supplied from the reservoir of liquid. As the exhaustion approaches the tension of aqueous vapour, evaporation goes on at a greater rate, and the vapour displaces the air with increasing rapidity; until, after the pressure of 12·7 millims. is passed, the aqueous vapour acts as a gas, and, being constantly supplied from the reservoir of water (as long as it lasts), washes out all the air from the apparatus, the logarithmic decrement rapidly sinking to that of pure water gas.

This explanation requires that the viscosity of pure aqueous vapour should be the same as that of hydrogen, at all events between 7 millims. and 1 millim. pressure. The facts can, however, be explained in another way. During the action of the Sprengel pump sufficient electricity is sometimes generated to render the fall tubes luminous in the dark (52). It is conceivable that under such electrical influence the falling mercury may be able to decompose aqueous vapour at these high exhaustions, with formation of oxide of mercury and liberation of hydrogen. Of these two theories the latter appears to be the more probable, but I have not sufficient data to enable me to decide between them.

703. The presence of water vapour shows itself likewise in the very slight amount of repulsion produced by radiation. Repulsion commences in air at a pressure of 12 millims., whilst at a higher exhaustion the maximum effect rises to over 40 divisions. Here, however, repulsion does not begin till the exhaustion is higher than the barometer gauge will indicate, whilst the maximum action after long-continued pumping is only 9 divisions. This confirms the results frequently met with in my researches on "Repulsion Resulting from Radiation," where the presence of even a trace

of aqueous vapour was found to have strong action in diminishing the sensitiveness of the radiometer and other instruments (130).

704. For a long time concentrated sulphuric acid was used to moisten the interior of the pump, a special stopper being affixed to the pump to admit of the acid being run in without interfering with the exhaustion. For six months observations were taken on air and several other gases, when it was found that the presence of sulphuric acid lowered the logarithmic decrement, and the work had all to be done over again. It is known, from the experiments of Messrs. Wanklyn and Robinson, that at a high temperature, and under conditions which admit of diffusion taking place, sulphuric acid dissociates into sulphuric anhydride and water; so it is not improbable that at a high exhaustion a similar decomposition may take place, one constituent being retained in the liquid acid more readily than the other, which escapes into the apparatus and alters the logarithmic decrement.

VISCOSITY OF KEROSOLINE VAPOUR.

705. The rapid diminution of viscosity in the last experiment after reaching the pressure of 400 millims. is probably due to the aqueous vapour in the air being near its liquefying point. It was thought advisable to test this hypothesis by employing a somewhat less easily condensible vapour, which could be introduced into the apparatus without any admixture of air. An experiment was accordingly tried with a very volatile hydrocarbon, commercially known as kerosoline, boiling at a little above the ordinary temperature. The vapour of this body was introduced into the well-exhausted apparatus, when the gauge at once sank 82·5 millims. After the usual precautions to eliminate air a series of observations were taken, with the following results:—

Table VIII.—Log. dec. of kerosoline vapour, at pressures between 82·5 millims. and 8 millims. Temp. 15° C.

Pressure in millims.	Log. decrement.
82·5	0·0425
71·5	0·0416
64·0	0·0409
59·0	0·0407
54·0	0·0404
48·5	0·0400
43·0	0·0396
38·0	0·0394
32·5	0·0389
21·0	0·0381
17·5	0·0380
13·4	0·0382
10·0	0·0381
8·0	0·0379

These results are plotted on the curve marked "Kerosoline Vapour," on diagram A. The loss of viscosity is more rapid than with any other gas examined except aqueous vapour. Conversely a very great increase of viscosity occurs on increasing the pressure from 8 to 82·5 millims. The explanation of this is that the vapour of kerosoline is very near its liquefying point, and therefore very far from the state of a "perfect" gas (709).

The negative bend in the curve at about 10 millims. pressure, already noticed with other gases (668), is strongly marked with this hydrocarbon vapour.

DISCUSSION OF RESULTS.

706. When discussing the viscosity results obtained with the different gases experimented with, I have given the proportion which I find between the viscosity of each gas and that of air, comparing the ratio with that obtained by GRAHAM, KUNDT and WARBURG, and MAXWELL, as follows :—

	GRAHAM.	KUNDT and WARBURG.	MAXWELL.	CROOKES.
Air	1·0000	1·0000	1·0000	1·0000
Oxygen	1·1099	..	.	1·1185
Nitrogen	0·971	..	..	0 9715
Carbonic anhydride. . .	0·807	0·806	0·859	0·9208
Carbonic oxide . . .	0·971	..	..	0·9715
Hydrogen	0·4855	0·488	0·5156	0·4439

I have reason to believe that my results are more accurate than those of other observers. In the case of hydrogen I have not only obtained a much lower viscosity, but the absolute obedience which it pays to MAXWELL'S law is an additional proof of its purity, for any admixture of foreign gas destroys the uniformity of results (688).

707. GRAHAM'S numbers are the theoretical results deduced from his experiments on transpiration of gases. They are, he says,* the numbers to which the transpiration times of the gases approximate and in which they have their limit. GRAHAM concludes that the "times of oxygen, nitrogen, carbonic oxide, and air are directly as their densities, or equal *weights* of these gases pass in equal times. Hydrogen passes in half the time of nitrogen, or twice as rapidly for equal volumes. The result for carbonic acid appears at first anomalous. It is that the transpiration time of this gas is inversely proportional to its density when compared with oxygen." In the Bakerian lecture already quoted, when discussing GRAHAM'S results and their bearings upon his own experiments on the viscosity of gases, MAXWELL says: "It appears to me that for comparative estimates of viscosity the method of transpiration is the best."

708. It must not be forgotten that the pressure of 760 millims. is not one of the constants of Nature, but is a purely arbitrary one, selected for our own convenience

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* *Loc. cit.*, pp 178–9.

when working near the level of the sea. In diagrams A, B, and C I have started from this pressure of 760 millims., and have given the curves through a wide range of exhaustion. But the curves might also be continued, working downwards instead of upwards. Although it is unsafe to indulge in speculation in the absence of data, there are some conclusions which it is quite legitimate to draw from an inspection of the curves. From the shape and direction in which they cut the 760 line it is reasonable to infer their further progress downwards, and we may assume that an easily liquefiable gas will show a more rapid increase in log. decrement than one which is difficult to liquefy by pressure. For instance, hydrogen, the least condensible of all gases, shows no tendency to increase in log. decrement by pressure. Oxygen and nitrogen, which are only a little less difficult to condense than hydrogen, show a slight increase in log. decrement. Carbonic anhydride, which liquefies at a pressure of 56 atmospheres at 15° C., increases so rapidly in viscosity that at this pressure it would have a logarithmic decrement of about 1·3, representing an amount of resistance to motion that it is difficult to conceive anything of the nature of gas being capable of exerting.

Kerosoline vapour is rendered liquid by pressure much more readily than carbonic anhydride. Its curve of viscosity on diagram A shows a great increase in density for a very slight access of pressure (705).

709. MAXWELL's law was discovered as the consequence of a mathematical theory. It presupposes the existence of gas in a "perfect" state—a state practically unknown to physicists, although hydrogen gas very nearly approaches that state. An ordinary gas may be said to be bounded, as regards its physical state, on the one side by the sub-gaseous or liquid condition, and on the other side by the ultra-gaseous condition. A gas assumes the former state when condensed by pressure or cold, and it changes to the latter state when highly rarefied. Before actually assuming either of these states there is a kind of foreshadowing of change, with partial loss of gaseity. When the molecules, by pressure or cold, are made to approach each other more closely, they begin to enter the sphere of each other's attraction, and therefore the amount of pressure or cold necessary to produce a certain density or viscosity is less than the theoretical amount by the internal attraction exerted on each other by the molecules. The nearer the gas approaches the point of liquefaction the greater is the attraction of one molecule to another, and the amount of pressure required to produce any given density will be proportionally less than that theoretically required by a "perfect" gas. This foreshadowing of the sub-gaseous or liquid state explains how it is that there is such wide divergence from MAXWELL's law in the case of imperfect gases, such as carbonic anhydride, water gas, and the volatile hydrocarbon kerosoline. At the other end of the scale we find even a more marked divergence from MAXWELL's law. This is due to the gas commencing to assume ultra-gaseous properties.

THE ULTRA-GASEOUS STATE OF MATTER.

710. A consideration of the curves of viscosity of the gases, especially hydrogen, which are given in the foregoing pages, will, I think, confirm the supposition that a gas, as the exhaustions become extreme, gradually loses its gaseous characteristics, and passes to what I have ventured to call an ultra-gaseous state. Certainly it ceases to possess many of the properties usually held to be the essential attributes of gaseity.

For instance, MAXWELL's law that the viscosity of a gas is independent of pressure holds good to a certain point, and then it rapidly breaks down. All gases appear to obey MAXWELL's law between some limits of exhaustion, and diverge from it at others. The change to the ultra-gaseous state commences to be assumed at about an exhaustion of half a millim. In hydrogen the change then proceeds slowly, but in the other gases I have experimented with the change to ultra-gas takes place with greater rapidity.

711. In gases, variation of pressure in different parts of a closed vessel equalises itself with great rapidity, but in the ultra-gaseous state differences of pressure may exist for twenty minutes or more in different parts of the apparatus.

712. In gases, electrically charged bodies do not permanently retain their charge, but gradually discharge themselves. In ultra-gas, however, a pair of electrified gold leaves have remained repelled at absolutely the same angle for thirteen months.*

713. Another property of gases is that of facilitating the cooling of bodies immersed in them, by communicating an increase of motion to the molecules of the gas which carry it to the walls of the containing vessel,—*i.e.*, by *carriage* instead of *convection.*

714. There is little difference in the rate of cooling with increased exhaustion, so long as we work with such ordinary good vacua as can be obtained by air-pumps. For if, on the one hand, there are fewer molecules impinging on the warm body (which is adverse to the carriage of heat), yet on the other hand, the mean length of path between collisions is increased so that the augmented motion is carried farther; the number of steps by which the temperature passes from the warmer to the cooler body is diminished, but the value of each step is correspondingly increased. Hence the difference of velocity before and after impact may make up for the diminution in the number of molecules impinging.

In gases, therefore, the rate of cooling is little affected by rarefaction, the law in this case being analogous to that governing the viscosity.

715. In a paper which I have recently had the honour of reading before the Royal Society,† I show that when the exhaustion is carried to so high a point that the mean free path is comparable with the diameter of the containing vessel, the rate at which heat is conveyed across is much diminished. The molecules are now in the ultra-gaseous state, and further exhaustion produces a notable fall in the rate of

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* Proc. Roy. Soc., No. 193, 1879, p. 347.

† Ibid., No. 208, 1880, p. 239.

cooling, an increase of exhaustion from 20 M to 2 M retarding the carriage of heat more than all the previous exhaustion from 760 millims. to 20 M.

716. I have shown elsewhere* that the property of gaseity is pre-eminently a property dependent on collisions. A given space full of air at the ordinary pressure contains millions of millions of molecules rapidly moving in all directions, each molecule momentarily encountering millions of other molecules in a second. In such a case the length of the mean free path of the molecules is exceedingly small compared with the dimensions of the containing vessel, and those properties are observed which constitute the ordinary gaseous state of matter—properties which depend upon constant collisions.

The gaseous state continues so long as the collisions are almost infinite in number and of inconceivable irregularity. But in such high vacua as I now describe the free path of the molecules is made so long that the hits in a given time may be disregarded in comparison to the misses, and the average molecule is allowed to obey its own motions or laws without interference; and when the mean free path is comparable to the dimensions of the containing vessel, the properties which constitute gaseity are reduced to a minimum, and the matter then becomes exalted to an ultra-gaseous state.

In the ultra-gaseous state properties of matter which exist even in the gaseous state are shown *directly*, whereas in the state of gas they are only shown *indirectly*, by viscosity and so forth.

717. The ordinary laws of gases are a simplification of the effects arising from the properties of matter in the ultra-gaseous state; such a simplification is only permissible when the mean length of path is small compared with the dimensions of the vessel. For the sake of simplicity we make abstraction of the individual molecules, and feign to our imagination *continuous* matter of which the fundamental properties—such as pressure varying as the density, and so forth—are ascertained by experiment. A gas is nothing more than an assemblage of molecules contemplated from a simplified point of view. When we deal with phenomena in which we are obliged to individually contemplate molecules, we must not speak of the assemblage as *gas*.

718. An objection has been raised touching the existence of ultra-gaseous matter in highly exhausted electrical tubes, that the special phenomena of radiation and phosphorescence which I have considered characteristic of this form of matter can be made to occur at much lower pressures than that which exhibits the maximum effects. For the sake of argument let us assume that the state of ultra gas with its associated phenomena is at the maximum at a millionth of an atmosphere. Here the mean free path is about 4 inches long, sufficient to strike across the exhausted tube. But it has been shown by many experimentalists that at exhaustions so low that the contents of the tube are certainly not in the ultra gaseous state, the phenomena of phosphorescence can be observed. This circumstance had not escaped my notice. In my first paper on the "Illumination of Lines of Molecular Pressure and the Trajectory of

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* Proc. Roy. Soc., No. 205, 1880, p. 469.

Molecules"* I drew attention to the fact that a molecular ray producing green phosphorescence can be projected 102 millims. from the negative pole when the pressure is as high as 0·324 millim. or 427 M. In this case the mean free path of the molecules is 0·23 millim.; and it is not surprising that with more powerful induction discharges, and with special appliances for exalting the faint action to be detected, the above-named phenomena can be produced at still higher pressures.

719. It must be remembered that we know nothing of the *absolute* length of the free path or the *absolute* velocity of a molecule; these may vary almost from zero to infinity. We must limit ourselves to the *mean* free path and the *mean* velocity, and all that these experiments show is that a few molecules can travel more than a hundred times the *mean* free path, and with perhaps a corresponding increase over the *mean* velocity, before they are stopped by collisions. With weak electrical power, the special phosphorogenic action of these few molecules is too faint to be noticed; but by intensifying the discharge the action of the molecules can be so increased as to render their presence visible. It is also probable that the absolute velocity of the molecules is increased so as to make the mean velocity with which they leave the negative pole greater than that of ordinary gaseous molecules. This being the case, they will not easily be stopped or deflected by collisions, but will drive through obstacles, and so travel to a greater distance.

If this view is correct, it does not follow that gas and ultra-gas can co-exist in the same vessel. All that can be legitimately inferred is, that the two states insensibly merge one into the other, so that at an intermediate point we can by appropriate means exalt either the phenomena due to gas or to ultra-gas. The same thing occurs between the states of solid and liquid, and liquid and gas. TRESCA'S experiments on the flow of solids prove that lead and even iron, at the common temperature, possess properties which strictly appertain to liquids, whilst ANDREWS has shown that liquid and gas may be made to merge gradually one into the other, so that at an intermediate point the substance partakes of the properties of both states.

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* Phil. Trans, Part I., 1879: The Bakerian Lecture.

Note on the Reduction of Mr. CROOKES's *Experiments on the Decrement of the Arc of Vibration of a Mica Plate oscillating within a Bulb containing more or less Rarefied Gas.*

By Professor G. G. STOKES, *Sec. R.S.*

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IN the course of his long series of researches "On Repulsion resulting from Radiation," Mr. CROOKES had frequently occasion to observe the deflections of a light bar or lamina of some substance delicately suspended and oscillating by torsion. When such a bar was set in vibration, the vibrations tended more or less rapidly to subside, in consequence, no doubt, of the viscosity of the gas enclosed in the apparatus. At first it seemed as if the rate of subsidence tended to reach a constant value which remained the same at all higher exhaustions. But as methods of exhaustion were improved, and the gases were so rarefied that the effect of a candle in causing repulsion distinctly fell off, the rate of subsidence of the oscillations was found greatly to fall off too. This falling off at extreme exhaustions seemed to present a very interesting field of study in connexion with the molecular condition of gases. The inquiry would naturally involve the observation of the nearly constant rate obtained at somewhat lower exhaustions; and the same apparatus would serve for experiments on the rate of subsidence at higher densities, up to that corresponding to atmospheric pressure.

A comparison of the rates of subsidence in different gases at great but not extreme exhaustions was further interesting as a new means of determining the ratios of the viscosities of different gases. In fact, at high exhaustions the motion of the gas tends to a condition of ideal simplicity from which a comparison of the viscosities of different gases would immediately result. The effect of the viscosity of a gas on its own motion is regulated by the value of a constant which I have elsewhere[*] called the index of friction of the gas, namely, the coefficient of viscosity divided by the density. According to MAXWELL's law, the coefficient of viscosity is independent of the density, and therefore the index of friction varies inversely as the density. Hence as the exhaustion proceeds the motion of the gas tends to become what it would be if the viscosity were infinite, and the bounding surfaces had their actual motion. In the limit, the instantaneous motion of the gas depends only on that of the vibrating plate, to which it is proportional, except in so far as the finiteness of the angle of oscillation entails a difference of position of the plate relatively to the bounding wall of the bulb, a difference however which is trifling on account of the smallness of the angle through which the plate oscillates. The forces therefore arising from the viscosity tend in the limit to vary entirely as the angular velocity, and not, as is the case when the index of friction is comparatively small, partly also on the angular acceleration. The result therefore will be that the oscillations are retarded by a force varying as the angular velocity, and producing therefore a subsidence of the motion such that the proportionate rate of change of the arc is proportional to the coefficient of viscosity.

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[*] "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums," Cambridge Philosophical Transactions, vol. ix., p. [8].

Mr. CROOKES's experiments were carefully made from pressures as high as the atmospheric pressure downwards. At first there is a very evident decrease of subsidence as the pressure decreases, except in the case of hydrogen, in which it is very small, we may say insensible. Then it remains very nearly constant for a considerable range of exhaustion, and at last, for extreme exhaustions, it rapidly fades away.

In the second of these three stages the condition of ideal simplicity above mentioned is doubtless approximately attained. If however we confined our attention to this part only of the series, the lower part, although so carefully made, would remain unutilised; and further, we should remain uncertain whether in taking the logarithmic decrement as proportional to the viscosity our approximation was not too rough.

The determination of the motion of the gas corresponding to a given motion of the vibrating solid, and thereby the determination of the forces which the gas exerts on the solid, forms a perfectly definite problem, the solution of which, if it could be effected, would lead to a determination of the coefficient of viscosity from the observed influence of the gas on the motion of the plate. But although in these slow motions the terms in the hydrodynamical equations which involve the squares of the velocities are insensible, so that the equations may be taken as linear, the problem is one of hopeless difficulty except in a few simple cases. In the paper referred to, I have given the solution in the case of a sphere vibrating in a mass of fluid either unlimited or confined by a concentric spherical envelope, and in that of a long cylindrical rod vibrating in an unconfined mass of fluid. In the latter especially of these cases the solution involves functions of a highly complicated form. For a lamina such as that employed by Mr. CROOKES, the problem could not even be solved if the fluid were regarded as perfect, much less when the viscosity is taken into account.

But though we are baffled in the attempt to give an absolute solution of the problem, theory indicates the conditions of similarity of the motion of the gas in two different cases, and enables us thereby to compare the viscosities when those conditions are satisfied.

The bulb, vibrating plate, and torsion thread being always the same, the two things that varied from one experiment to another were the nature and the pressure of the gas, not the temperature, which for the present was kept constant. The moment of inertia of the lamina was sufficiently large to allow the time of vibration to be nearly the same in the different experiments. For the present I will suppose it constant, reserving to a later stage the consideration of the correction to be made for its variation.

Let ρ be the density of the gas, p the observed pressure, D the density under a standard pressure, μ the coefficient of viscosity, and let accented letters refer to another gas. The dimensions of the terms in the equations of motion show that in comparing two cases in which the nature and pressure of the gases alone differ, the motions will be similar provided

$$\frac{\mu}{\rho}=\frac{\mu'}{\rho'}, \text{ or } \frac{\mu}{p\mathrm{D}}=\frac{\mu'}{p'\mathrm{D}'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This condition being satisfied, the resultant pressures of the gas on the solid will vary as μ or as ρ; and as the logarithmic decrements (l) will vary in the same proportion, we shall have

$$\frac{l}{\rho}=\frac{l'}{\rho'}, \text{ or } \frac{l}{p\mathrm{D}}=\frac{l'}{p'\mathrm{D}'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The equations (1) and (2) are such that when one is satisfied so is the other. It will be convenient to regard (2) as giving the condition of similarity, and then (1) or

$$\frac{\mu}{l}=\frac{\mu'}{l'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

gives the ratio of the viscosities at the two corresponding pressures in the two gases.

The times of vibration were practically constant when once the exhaustion was pretty high, at least until the very highest exhaustions were reached, when it fell off a very little; but at atmospheric pressure and at low exhaustions it was somewhat greater, though not much. Its variability will not affect the results obtained by the above method, provided only the times are the same in the two experiments of each pair, which was very approximately the case. Nevertheless it may be well to consider the correction to be made in consequence of the inequality of the times.

Let τ be the time of vibration from rest to rest, then in comparing two similar systems the time-scale must be varied, so as always to be proportional to τ, and the hydrodynamical equations show that for the condition of similarity we have, in place of (1), the equation

$$\frac{\mu\tau}{\rho}=\frac{\mu'\tau'}{\rho'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As the two dynamical systems are not similar as a whole, but only the gaseous parts of them, we must have recourse to the equation of motion of the vibrating lamina. Let θ be the angle of torsion, I the moment of inertia, $n^2\mathrm{I}\theta$ the force of restitution, which will be proportional to the angle of torsion, provided at least the glass fibre be treated as perfectly elastic, as it doubtless may be in considering the correction to be made for the inequality of the times of vibration, even though its defect of elasticity might not, possibly, be absolutely insensible in its influence on the main motion. Then if there were no fluid the equation of motion of the lamina would be

$$\mathrm{I}\frac{d^2\theta}{dt^2}+n^2\mathrm{I}\theta=0.$$

As in the cases treated of in the paper of mine already referred to, the resultant pressure of the fluid on the lamina (the term "pressure" here including the tangential action), will partly agree in phase with the displacement or force of restitution, partly with the velocity of the lamina. The first part will have the effect of adding to the mass of the lamina an ideal mass depending on the density, viscosity, and time of vibration. From the dimensions of the quantities involved with respect to time and density, this ideal mass must be of the form $\rho f\left(\frac{\mu\tau}{\rho}\right)$. There is no need to express the dependence of the function f on the scale of lengths. In a similar manner the part of the resultant pressure which is multiplied by $\frac{d\theta}{dt}$ must be expressed by ρ multiplied by some other function of $\frac{\mu\tau}{\rho}$ and divided by a time. We may express it therefore by $2\rho n\mathrm{F}\left(\frac{\mu\tau}{\rho}\right)$ where n is $\frac{\tau}{\pi}$. Denoting the two functions of $\frac{\mu\tau}{\rho}$ by A and B respectively, we have accordingly for the equation of motion

$$(\mathrm{I}+\mathrm{A}\rho)\frac{d^2\theta}{dt^2}+2\mathrm{B}\rho n\frac{d\theta}{dt}+a^2\mathrm{I}\theta=0\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The integral of this equation is

$$\theta=e^{-qt}(c\cos mt+c'\sin mt)$$

where

$$q=\frac{\mathrm{B}\rho n}{\mathrm{I}+\mathrm{A}\rho},\ m^2=\frac{a^2\mathrm{I}}{\mathrm{I}+\mathrm{A}\rho}-\frac{\mathrm{B}^2\rho^2n^2}{(\mathrm{I}+\mathrm{A}\rho)^2};$$

and since by definition of n, $m=n$, we have

$$n^2+q^2=\frac{a^2\mathrm{I}}{\mathrm{I}+\mathrm{A}\rho},$$

and then by eliminating A between the last equation and the last but two we have

$$\mathrm{B}=\frac{a^2\mathrm{I}q}{\rho n(n^2+q^2)}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now n is $\pi \div \tau$, and $q\tau$ is the Napierian logarithmic decrement, or $l \div \mathrm{M}$, M being the modulus of the common system. Hence (6) becomes

$$\mathrm{B}=\frac{a^2\mathrm{M}\,\mathrm{I}l\tau^2}{\pi\rho(\mathrm{M}^2\pi^2+l^2)}$$

and as B is the same in the two systems compared we have

$$\frac{l}{\rho}\cdot\frac{\tau^2}{\mathrm{M}^2\pi^2+l^2}=\frac{l'}{\rho'}\cdot\frac{\tau'^2}{\mathrm{M}^2\pi^2+l'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

an equation which takes the place of (2), and serves to define corresponding densities, and then (4) gives the ratio of the viscosities at those densities, or say, at the corresponding pressures. If we eliminate the ratio of ρ to ρ' between (4) and (7) we get

$$\frac{\mu}{l\tau}(\pi^2\mathrm{M}^2+l^2)=\frac{\mu'}{l'\tau'}(\pi^2\mathrm{M}^2+l'^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which takes the place of (3).

The ratio of the factor $\pi^2\mathrm{M}^2+l^2$ to $\pi^2\mathrm{M}^2$ alone but little exceeds unity; thus even for oxygen at 760 millims. pressure it is barely 1·0085, and of course the ratio of that factor for one gas to the factor for another gas at the corresponding pressure will differ from unity still less. Hence it is almost a needless refinement to keep in this factor at all. However, even if we retain it in (8) it is quite superfluous in (7), which merely determines what densities are to be deemed to correspond in seeking the logarithmic decrements. For until extreme rarefactions are reached, to which the above investigation no longer applies, the logarithmic decrement changes so slowly that a small error in the density of one gas which is deemed to correspond to a given density in another will make no sensible error in the logarithmic decrement. And not only may the factors above mentioned be omitted, but as the ratio of τ to τ' will differ but little from a ratio of equality, the formula (7) may be dispensed with altogether, and the simpler formula (2) employed. But when the logarithmic decrements have been found, in determining the ratio of the viscosities from (8) it is better not to disregard the quantities by which the ratios of τ to τ' and of $\pi^2\mathrm{M}^2+l^2$ to $\pi^2\mathrm{M}^2+l'^2$ differ from ratios of equality. And if we now wish to know more precisely what densities or pressures do correspond, we may obtain them from (4).

In the numerical calculations which follow, the difference in the times of vibration (τ) at *corresponding* pressures in the different gases is neglected, and likewise the difference between the ratios of the factors $\mathrm{M}^2\pi^2+l^2$ and a ratio of equality. The general effect of this omission, which is very minute, will be considered in the end.

The values of D adopted were, hydrogen, 1; air, 14·42; oxygen, 16; nitrogen, 14; carbonic anhydride, 22; carbonic oxide, 14.

Tables were first formed of the logarithms of $\frac{l}{pD}$ for the different gases for the various pressures given by Mr. CROOKES down to 0·76 millim. The pressures standing against equal numbers in these tables for the different gases would be "corresponding" pressures. The pressures corresponding to a given number may be obtained from the tables by interpolation; and as the experiments were made at close intervals it seemed sufficient to regard only the two adjacent numbers and use proportional parts.

I do not think it worth while to give these tables at length, but I subjoin a small table calculated from them, giving for oxygen, nitrogen, carbonic anhydride, and carbonic oxide the pressures corresponding to air pressures decreasing by 100 millims. from 760 to 160. It will be seen from Mr. CROOKES's tables that below these pressures there is little variation in l until very high exhaustions are reached. Hydrogen does not enter into the table, as the highest pressure (760 millims.) in the experiments corresponds to a pressure of only about 106 millims. in air. The table contains also the logarithms of the ratios of the pressures in the different gases to the corresponding pressures in air.

Corresponding pressures.					Logs. of ratios to corresponding pressures in air.			
Air.	O.	N.	CO_2.	CO.	O.	N.	CO_2.	CO.
760	767·3	760·8	413·3	760·8	+ ·0042	+ ·0005	− ·2645	+ ·0005
660	666·6	660·9	359·2	663·0	+ ·0044	+ ·0006	− ·2642	+ ·0020
560	565·5	559·6	306·2	563·8	+ ·0042	− ·0003	− ·2622	+ ·0029
460	463·7	459·0	250·0	459·8	+ ·0034	− ·0010	− ·2649	− ·0002
360	365·9	361·6	194·7	359·5	+ ·0071	+ ·0019	− ·2669	− ·0006
260	263·2	261·8	141·9	258·3	+ ·0053	+ ·0030	− ·2630	− ·0029
160	161·2	159·5	86·4	166·7	+ ·0033	− ·0013	− ·2676	− ·0022
Mean of log. p'/p					+ ·0046	+ ·0005	− ·2648	− ·0005
Number, or p'/p.					1·010	1·001	0·540	0·999
Log. ($p'D' \div pD$), from mean . . .					·0497	$\bar{1}$·9876	$\bar{1}$·9186	$\bar{1}$·9870
$p'D' \div pD$					1·121	0·972	0·829	0·971

An inspection of the numbers in the same vertical column in the right-hand portion of the above table shows that the logarithm in question is constant as nearly as the observations can show. This leads to the following law.

If any pressure be taken in one gas and the pressures found in other gases for which the coefficients of viscosity are as the densities (pressures which have been defined as "corresponding"), then if another system of pressures be taken proportional to the former the pressures in the new system will also correspond; and consequently the ratios of the coefficients of viscosity of the different gases will be the same for the pressures in one such system as in another.

This law is in accordance with MAXWELL's law, but does not by itself alone prove MAXWELL's law. It leaves the functional relation between the coefficient of viscosity and the density for any one gas arbitrary, and deduces from thence the relation for all other gases, this relation introducing merely one unknown constant for each. What the law gives may be put in a clear form by a geometrical illustration. I assume BOYLE's law, so that for any one gas the ratios of the densities in different cases or the ratios of the pressures may be used indifferently.

Let, then, the relation between the viscosity and density be represented graphically by taking abscissæ to represent densities and ordinates to represent coefficients of viscosity. Then the law found above may be enunciated by saying that the curves for all gases are geometrically similar, the origin being the centre of similitude. MAXWELL's law would give a particular case of such similar curves, namely, a system of straight lines parallel to the axis of abscissæ.

The deviation from uniformity of the logarithmic decrements for any one of these gases at these comparatively speaking high pressures is not therefore in any way inconsistent with MAXWELL's law, but is fully accounted for by the very natural supposition that the rarefaction is not yet sufficient to render the molar inertia of the gas insensible as regards its influence on the gas's own motion, a supposition which can be shown to be true when we employ the approximately known absolute value of the coefficient of viscosity. The same consideration shows, moreover, that we have only to carry the exhaustion further in order to render the effect of that inertia insensible, and accordingly, *if* MAXWELL's *law be true*, to make the logarithmic decrement sensibly independent of the pressure.

That such is actually the case is shown by Mr. CROOKES's tables or the diagram A, in which they are graphically represented. We observe a manifest tendency for the logarithmic decrement to become constant till the law is interrupted by the breaking down of viscosity attending extreme exhaustions, or by certain deviations which in some cases (as in those of oxygen and kerosoline vapour) show themselves a little earlier: these deviations will be referred to further on; for the present I merely avoid exhaustions high enough to introduce them. This approximate constancy of logarithmic decrement is observed in hydrogen from the first, which is accounted for by the high index of friction of that gas as compared with the others at equal pressure.

This evident constancy or tendency towards constancy in the viscosity as the rarefaction goes on supplies the missing link, and establishes MAXWELL's law on the basis of Mr. CROOKES's experiments even taken by themselves. It is not, of course, *directly proved* for the higher pressures in the gases other than hydrogen; its extension to such pressures is a matter of inference, derived from observing, first, that it is found to be true within such limits of density that the condition of ideal simplicity supposed at the commencement of this note is presumably sensibly attained; and, secondly, that above those limits, though we are unable from mathematical difficulties to examine its truth directly, yet we are able to deduce from theory one inference on the supposition of its truth, which is found to be in accordance with the results of experiment.

Hitherto the ratios of the coefficients of viscosity have been deduced from a part of Mr. Crookes's tables, in which the logarithmic decrements changed very evidently with the pressure. We may now deduce those ratios by what may almost be deemed an independent method, namely, by attending only to the part of the tables at which the logarithmic decrement is all but constant. If the condition of ideal simplicity supposed at the outset were quite attained, we might disregard the pressures in the comparison, which would entitle that method to be considered quite distinct from the former; but as that condition is not absolutely reached, it will be proper not wholly to neglect the condition of correspondence of pressure, though a rough determination of correspondence will suffice.

Suppose, then, we take the air pressures from 120 to 26 millims. The ratio of the corresponding pressures in oxygen, &c., is given in the last line but two of the preceding table. The corresponding limits are for oxygen 132 to 29; for nitrogen and carbonic oxide the same (sensibly) as for air; for carbonic anhydride 65 to 14; for hydrogen the limits are not given in the table, but they are 864 and 187 nearly. In strictness each pressure should be considered separately; but as the intervals were not intentionally divided in a different manner for the different gases, and as the logarithmic decrements are very nearly constant between the specified limits, it seems sufficient to take the mean for each gas of those corresponding to pressures that lie between the assigned limits. We thus get for air, ·1002; oxygen, ·1120; nitrogen, ·971; carbonic anhydride, ·822; carbonic oxide, ·971; hydrogen, ·500. Reducing to air $=1$, and writing down for comparison the numbers expressing the ratios of the coefficients of viscosity to that of air given in the previous table, we have for the ratios in question—

	O.	N.	CO_2.	CO.	H.
From air pressures, 760 to 160 .	1·121	0·972	0·829	0·971	..
„ „ 120 to 26 .	1·118	0·969	0·820	0·969	0·499
Values adopted	1·120	0·970	0·822	0·970	0·499

We see that almost identically the same numbers are obtained whether they are deduced from the higher pressures, for which the logarithmic decrements notably diminish with the pressure, or from the part of the tables in which they are nearly independent of the pressure. The greatest difference is in the case of carbonic anhydride, where it is rather more than one per cent. This difference is in part accounted for by the omission of the correction for the time of vibration. If the times of vibration at *corresponding* pressures as determined by Mr. Crookes be taken, they will be found to be very nearly the same; indeed, the differences are quite comparable with the errors of observation of those times. Perhaps the differences could be got with greater certainty from theory than from observation. According to theory the effect of the gas on the time of vibration turns mainly on the term $A\rho$. Now, though A is a function which we cannot calculate, yet we know that it is the same at corresponding pressures in two gases. The effect at such pressures varies therefore from one gas to another as the density, and therefore as the coefficient of viscosity. It is here supposed (as is practically true) that the term is so small compared with I, with which it is associated, that its square, &c., may be neglected.

Taking the time of two oscillations (or of one complete oscillation) for air at an exhaustion at which the effect of the molar inertia has ceased to be sensible, but the slight decrease due to the removal of the viscosity has not yet come in, at $10^{\mathrm{s}}{\cdot}76$, we get for the mean effect at pressures 760,660 . . . 160 about $0^{\mathrm{s}}{\cdot}20$. The coefficient of viscosity for carbonic anhydride being 18 per cent. less than for air, we get $0^{\mathrm{s}}{\cdot}036$ for the average difference of times in air and that gas, which is $\frac{1}{300}$th of the average time; and since according to (8) $\mu \propto \tau$ we must deduct ·003 from the ·829 given above, leaving ·826, the mean of which and ·820 gives ·823, nearly the number adopted. Similarly the number 1·121 for O in the first line should be raised about ·002.

There is still a small correction to make depending on the factor $\pi^2M^2+l^2$. Since by (8) μ varies as this factor, and l^2 is very small compared with π^2M^2, and moreover $\mu \propto l$ nearly, it will suffice to deduct $\frac{l^3}{\pi^2M^2}$ from l, and use the l's so corrected. The correction being, however, very small, it will suffice to take an average l and make the deduction for it. The deductions for the six gases came to about ·005, ·009, ·005. ·003, ·005, ·001. Deducting these numbers from the relative viscosities given above, and reducing afresh to the scale air $=1$, we get the following final numbers:

Air.	O	N	CO_2	CO	H
1·000	1·117	0·970	0·823	0·970	0·500

I have left kerosoline vapour to the last on account of the uncertainty as to its vapour density. It is a mixture of different substances, being the more volatile part of petroleum. I am informed by Mr. GREVILLE WILLIAMS that it contains much pentane, the theoretical vapour density of which on the hydrogen scale would be 36. Taking at a venture $D=36$, and choosing suppose the pressure 54 millims., for which $l={\cdot}0404$, and further assuming the limiting logarithmic decrement for air before the breakdown to be 0·1000, as it seems to be from Mr. CROOKES's table, we find 0·392 for the relative viscosity of kerosoline vapour. This is pretty certainly too high. If we suppose the true number to be 0·380, we get for the air number corresponding to $l'={\cdot}0425$, $l={\cdot}1129$, which from the table of results for air belongs to $p=740$. This would give for the vapour density of kerosoline $D'=\frac{{\cdot}0380\times 740}{{\cdot}1000\times 82{\cdot}5}D=3{\cdot}408D=49{\cdot}16$ if $D=14{\cdot}42$.

For the air pressure corresponding to 54 millims. in kerosoline we should have $740\times 54\div 82{\cdot}5=484{\cdot}3$. For the corresponding logarithmic decrement we get from the table of experiments for air ·1059. The logarithmic decrement for kerosoline at 54 millims. calculated from that for air at 484·3 millims., is ·1059 × ·0390 = ·04024, which comes very near the observed number ·0404. It is curious that we should apparently be able to calculate very approximately an unknown vapour density from observations on the decrement of the arc of vibration of a vibrating lamina.

Before considering the falling off of viscosity at high exhaustions, I would point out a result of theory which is of some interest in connexion with the forms of Mr. CROOKES's curves.

At p. [34] in the paper of mine already referred to I have given in equation (61) an expression (which has to be subsequently multiplied by ρ) for the resistance to a sphere vibrating in a viscous fluid within a concentric spherical envelope. When the index of friction is sufficiently great, as will be the case when the exhaustion is high enough, this expression may be developed according to ascending powers of m, which will be convergent even from the first. It will be found that the successive terms will be multiplied by

$$m^{-2},\ m^0,\ m^2,\ m^4\ \dots$$

where m^2 is a pure imaginary multiplied by $\rho \div \mu$; and from the mode of treatment there adopted it will readily be seen that the terms fall alternately on the arc and on the time.

The same thing may, however, be shown to be true generally, independently of the form of the vibrating body. It is here supposed, as has been all along, that the motion is sufficiently slow to allow us to neglect squares and products of the components of the velocity, or of their differential coefficients. For if $\mu \div \rho$ be very great, we may imagine the hydrodynamical equations solved by successive substitution. First we should neglect the terms in ρ, and solve the equations; then substitute in the terms multiplied by ρ the result of the first approximation and solve again, and so on. And though we cannot actually solve the equations, still this consideration shows that the solution must be of the form

$$a\mu + b\rho + c\frac{\rho^2}{\mu} + d\frac{\rho^3}{\mu^2} + \dots$$

where $a, b, c, d \dots$ involve neither μ nor ρ. And by adopting the artifice for the introduction of the time employed in the paper it readily appears that the terms fall alternately on the arc and on the time. Hence taking the two most important terms only in the expression for the effect on the arc we shall have for l an expression of the form

$$a'\mu + c'\frac{\rho^2}{\mu}, \text{ or } a'\mu + c'\frac{D^2}{\mu}p^2.$$

Hence in a curve plotted with l and p for coordinates, the tangent as p diminishes would tend to become parallel to the axis of p, and the curvature where the curve cuts the axis of l would vary from one gas to another directly as the square of the density at a standard pressure and inversely as the coefficient of viscosity. For the gases examined the curvatures near the axis of l would therefore vary

for	Air	O	N and CO	CO_2	H
as	$14{\cdot}42^2 \div 1{\cdot}000$,	$16^2 \div 1{\cdot}117$,	$14^2 \div 0{\cdot}970$,	$22^2 \div 0{\cdot}823$,	$1^2 \div 0{\cdot}500$,
or as	1·000,	1·102,	0·971,	2·828,	0·010.

It will be seen in Mr. CROOKES's diagram A, that if we imagine the curves continued upwards on their old lines, cutting off the changes which take place at very high exhaustions, the tangent tends to become vertical, and, moreover, the rate at which the direction of the tangent changes as we go down agrees well, as far as the eye can judge, with the above figures. We may notice in particular the extreme flatness of the hydrogen curve.

As we proceed upwards to the higher exhaustions, the first thing that strikes us (first in order of occurrence, very far from first in order of magnitude) is the curious *in*crease which is observed in the logarithmic decrement in the case of oxygen and of kerosoline vapour. Small as this is, Mr. CROOKES considers it real. It puzzled me at first, since it occurs while the pressure is still comparatively high, such as 15 millims. or 20 millims., so that the mean free path must still be extremely small, and might, one would naturally suppose, be treated as infinitely small considering the dimensions of the apparatus. It occurred to me afterwards that it is probably referable to the thinness of the vibrating body. As the lamina moves there must in the immediate neighbourhood of the edge be a thin stratum or cushion of gas in which there is a very intense shearing motion. The intensity of the shearing makes up in good measure for the narrowness of the cushion, and renders the effect of the cushion a not insignificant fraction of the whole. The narrowness of the stratum may well be such as to forbid us to treat the mean free path as infinitesimal long before we are prohibited from so regarding it in comparison with the dimensions of the vessel, or the lateral dimensions of the lamina. That among the four unmixed gases examined oxygen should be the one to show this effect, seems to be connected with the fact that at comparatively low exhaustions (such as 0·76 millim.) it shows repulsion effects much the most strongly; and both phenomena seem to indicate that for oxygen the length of path (I do not say *free* path) is comparatively large, "path" here meaning the space throughout which a molecule preserves approximately its original direction of motion.

When we come to those high exhaustions at which the decrement of arc gives way, the law of proportional logarithmic decrements at corresponding (and those proportional) pressures, which we hitherto found to be so accurately obeyed, breaks down altogether. A single example will suffice to show this. Take hydrogen at 330 **M**, for which $l=0{\cdot}0495$. The ratios of corresponding pressures resulting from the numbers for the relative viscosities given on page 443, and the numbers to which they lead, are as follows :—

Gas.	Air.	O.	N.	CO_2.	CO.	H.
Ratios of corresponding pressures . .	0·1387	0·1396	0·1386	0·0748	0·1386	1
Pressures corresponding to 330 M in H .	45·8	46·1	45·7	24·7	45·7	330·0
l calculated from l in H	·0990	·1106	·0961	·0815	·0961	
l observed	·0758	·0829	·0722	·0525	·0734	0·0495

It would seem as if when a gas may be treated as a continuous mass with continuously varying conditions of pressure, velocity, &c., as is done in the application of the hydrodynamical equations, a gas is completely defined* as to its mechanical action by two constants, suppose the density at a standard pressure and the coefficient of viscosity; but when the conditions are such as oblige us to take account of the finiteness of path of the molecules, specific differences are manifested which oblige us to introduce at least one constant more in order that the gas may be even mechanically defined; for of course I am not contemplating the chemical properties. It is worthy of note in this connexion that the two gases, oxygen and kerosoline vapour, which showed the phenomenon of a rate of decrease of arc of vibration *in*creasing with a decreasing density, are just those which lie at the two extremes as regards viscosity, while as regards density at a given pressure they are separated by carbonic anhydride, which nevertheless does not show the phenomenon in question. It may well be that the mode of encounter of such complex structures as the molecules of a gas varies from one gas to another; and that while some of the laws of gases admit of explanation when the molecules are regarded as elastic spheres, or as particles repelling one another according to some definite law of force, other properties fail to receive an explanation when such a simplification of conception is adopted.

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* I here leave out of account such differences as the small deviations from BOYLE's law which have been observed with different gases.

VIII. *On the Electrical Resistance of Thin Liquid Films, with a revision of* NEWTON'S *Table of Colours.*

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[PLATES 59, 60.]

I.

IN a preliminary note read before the Royal Society in June, 1877,* we stated the results of some observations in which an electric current was used in investigating the properties of soap films. These observations were made in the course of an enquiry (then incomplete) as to whether the resistance offered by a soap film to an electric current is inversely proportional to its thickness, and our object in undertaking this was to obtain, by a novel method, evidence as to the value of previous experiments by which various physicists had from time to time attempted "to obtain from the phenomena of capillarity, or from observations on liquid films, an indication of the magnitude of the radius of molecular attraction."

Since that date we have had but few opportunities of carrying out our research in common (as was necessary), and hence the long delay which has taken place; but we are now in a position to state the results of our later experiments, in which the method has been in several respects altered, and the apparatus considerably improved.

In the first place, it will be well to give a summary of the conclusions of previous observers. QUINCKE,† as the result of a well-known research, conducted by immersing in water and mercury plates of glass covered by wedge-shaped films of silver, collodion, and other convenient substances, arrived at the conclusion that the radius of molecular attraction is approximately equal to 50×10^{-7} centims. PLATEAU‡ estimates it at less than 59×10^{-7} centims. LÜDTGE,§ on the other hand, asserts that the thickness of a soap film becomes almost immediately after its formation less than twice the radius of molecular attraction. This conclusion, if correct, would show that QUINCKE's number is at least ten times too small.

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* Proc. Roy. Soc., No. 182, 1877.

† Pogg. Ann., 1869, Bd. cxxxvii., p. 402.

‡ 'Statique des Liquides,' 1873, tom. i., p. 210.

§ Pogg. Ann., 1870, Bd. cxxxix., p. 620.

The values derived from the dynamical theory of gases are, however, much smaller. VAN DER WAALS[*] finds $0{\cdot}23 \times 10^{-7}$ centims. to be an inferior limit, but considers QUINCKE's value too high.

O. E. MEYER,[†] on the contrary, thinks that the cause of the discrepancy is to be found in the fact that the expression "radius of molecular attraction" (*Wirkungssphäre*) is used in the theory of capillarity in a sense different from that in which it is employed in the theory of gases. He believes that, if we possessed a kinetic theory of liquids, the magnitude investigated by QUINCKE and PLATEAU would be found to be that of the free path of the molecules in the liquid, or would acquire some similar meaning.

PLATEAU's experiments consisted in measuring the pressure of the air confined in a soap bubble by means of a water manometer. A decrease in that pressure might be expected if the thickness of the film became less than twice the radius of molecular attraction. His conclusion is based on the fact that no regular change was observed in the case of a bubble which lasted for three days, and thinned to the pale yellow of the first order of NEWTON's scale.

LUDTGE formed a liquid film at one end of a cylindrical tube, and, when it had become thin, closed the other end by a newly formed film. A small quantity of air was then forced into the enclosed space through an orifice in the side of the tube. The films assumed the form of spherical segments, and it was found that that produced by the thinner film was the less curved. The inference that the superficial tension increased as the thickness of the film diminished, was supported by other experiments, for an account of which we must refer to the original paper. We may, however, remark that the large magnitude assigned to the radius of molecular attraction, and the statement that the superficial tension is increased instead of diminished when the thickness of a film is less than twice that magnitude, are opposed to the views of most physicists.[‡] The observations both of PLATEAU and LUDTGE are open to the objection that there is no proof that the soap solution used remained unaltered during the experiments.

A mixture of glycerine and water will gain or lose water according to the magnitude of the tension of the aqueous vapour in its neighbourhood. The superficial tension of a soap solution would thus tend to increase in damp, to decrease in dry, air. The direct effects of absorption or evaporation might, however, be diminished or even reversed by the changes of temperature with which they were accompanied. PLATEAU, to prevent the absorption of moisture, enclosed the bubble in a small glass vessel, at the bottom of which were placed some sticks of caustic potash. This arrangement would tend, and in fact was designed, to reduce the thickness of the film by evaporation, and would thus alter its constitution.

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[*] 'Beiblätter,' Bd. i., p. 19.

[†] 'Die Kinetische Theorie der Gase,' 1877, p. 236.

[‡] See 'Statique des Liquides,' PLATEAU, tom. i., p. 205, 1873. Also 'Théorie Mécanique de la Chaleur,' DUPRÉ, p. 358, 1867.

LUDTGE, anticipating the objection that the effect observed may have been due to the larger proportion of water absorbed by the older and thinner film, points to similar results obtained with a decoction of Quillaja. He appears to have overlooked the fact that if the experiments were performed in a dry atmosphere, the cold produced by evaporation might have been the true cause in both cases.

The difficulty of estimating accurately what the magnitude of the effect of such a cause might be was much enhanced by the fact that no observations had been made on the amount of water gained or lost by a soap film in a given time. The retrogression of the colours, which, after indicating for some time a decrease, often show an increase in thickness, had indeed been observed by PLATEAU. It was, however, impossible to say how far this change was directly due to absorption. Considerable masses of liquid are, as we shall hereafter show, often drawn into a film from the reservoir furnished by the relatively thick portions in contact with its solid supports. An imperceptible current of this kind might, whatever its cause may be, increase the thickness of the film without altering the proportions of its constituents. On the other hand, films thinning under the influence of gravity might, and, as our observations prove, often do, absorb large quantities of vapour without any sign of retrogression in the colours. Under these circumstances, the results of previous observers being in conflict, and so little being known as to the magnitude and rapidity of the changes of constitution which a soap film could undergo without rupture, it appeared desirable to investigate the subject from a novel point of view.

SONDHAUS* had attempted to study the passage of an electric current through a soap film, but failed, as his apparatus was not sufficiently delicate. With this exception, no investigation had, so far as we are aware, been made previous to our own on the resistance of liquid films. Experiments of this kind nevertheless seemed likely to afford information as to the probability of gaining, from observations on liquid films, any accurate knowledge of the magnitude of the radius of molecular attraction. In the entire absence of experimental results it was doubtful whether the specific resistance of a film is the same as that of the liquid from which it is produced. If the magnitude of the radius of molecular attraction is as great as is supposed by LÜDTGE or even by QUINCKE, some indication of the approach of a film to double that magnitude might be given by a change in its mean specific resistance. On the other hand, something might be learnt by properly contrived experiments as to the alterations in the constitution of a film wrought by variable hygrometric conditions.

The object then of our investigation has been to trace the effect on the specific resistance of a soap film—

(1.) Of change of thickness;

(2.) Of change in the hygrometric condition of the surrounding air.

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* Pogg. Ann. 1876, Bd. clvii., p. 95.

By comparing the results with observations on the liquid in mass, we are able to draw certain conclusions as to the light thrown on the molecular constitution of liquids by experiments on soap films.

The investigation divided itself into two parts, viz.: the determination (1) of the thickness, (2) of the resistance, of a film at any particular epoch, and as these, though carried on at the same time, were to a great extent independent, it will be convenient at first to treat of the former alone.

II. *The Liquid.*

The soap films were made in accordance with the receipt given by M. PLATEAU, with the exception that a small quantity of saltpetre was added to increase the conductivity. One part by weight of oleate of soda was dissolved in 40 parts of water, together with (in general) either 3, 5 or 7 parts of saltpetre to 100 of water. Three parts by volume of this liquid were then mixed with 2·2 parts of PRICE's glycerine. It will be convenient to call a solution containing *n* parts of saltpetre to 100 of water an *n* per cent. solution. The solutions thus prepared produced in the course of a few days a flocculent precipitate which could not be separated by filtering. They were therefore placed in long vertical glass tubes furnished with caoutchouc stoppers at the upper, and stop-cocks at the lower ends. In time the liquids cleared by the precipitate rising to the top, and became sufficiently transparent to allow of their refractive indices being measured. They could then be drawn off for use without disturbing the impurities on the surface. This operation of clearing occupied several months, and was always carefully performed in order to ensure the homogeneity of the liquids. It will be convenient to refer to a liquid thus prepared as a *standard solution;* others containing greater or less proportions of glycerine will be called *derived solutions.*

III. *Refractive Index.*

The following table (Table I.) gives the refractive indices, for sodium light, of several solutions, the composition of which is indicated in the first three columns.

TABLE I.

Percentage of salt in standard solution.	Parts of standard solution by volume.	Parts of water or glycerine by volume.	Refractive index.
5	100	0	1·3969
5	95	5 H_2O	1 3947
5	95	5 glycerine	1·4002
. .	0	100 glycerine	1·4715
8·82	75·6	24·4 glycerine	1·4155

The refractive index no doubt varied from film to film, as the composition of the liquid from which they were formed was often considerably altered by the absorption or evaporation of water. As a knowledge of the amount of water thus gained or lost was one of the objects in view, it was necessary to assume as a first approximation that the refractive index was constant. The method of testing and improving upon this assumption will be explained hereafter.

IV.—*Revision of* NEWTON'S *Table of Colours.*

The determination of the thickness of the films involved a knowledge of (in addition to the refractive index) (1) the angle of incidence, (2) the colour of the film, (3) the thickness of a plate of air exhibiting that colour when illuminated by light at normal incidence. . The method of measuring the angle of incidence will be described when an account is given of the apparatus employed (see p. 461). It is sufficient to say here that the films experimented upon were in the form of cylinders with vertical axes. Light was reflected upon them from a fixed mirror; a brightly illuminated vertical band was thus produced, and the thickness at any point was determined by the colour exhibited by this band. To increase and test the accuracy of the observations, two mirrors and therefore two beams of light, incident at different angles, were used.

The apparent thickness of a film is defined to be the thickness of a plate of air which shows, when illuminated at normal incidence, the same colour as that displayed by the more obliquely illuminated of the two bands. A good deal of labour was saved in the calculations by the use of this quantity instead of the real thickness, which could of course be readily deduced from it.

NEWTON, in describing the rings which bear his name, gives two lists of colours. In the first* seventeen colours are enumerated in the first four orders. In the second, which is reproduced in WATTS' 'Dictionary of Chemistry' (Art. *Light*), this number is increased to twenty-six. In this more extended list many tints are included which shade into those nearest to them by gradations too subtle to be readily appreciated; and since for purposes of measurement it is of little use to retain the names of colours unless the eye is able to distinguish with some approach to certainty where they begin and end, the following list, but little more extended than NEWTON'S first, was employed. The colours in brackets are those which are not included in NEWTON'S first list. The blue of the first order is omitted.

First order: Black, white, yellow, [orange], red.

Second order: Voilet, blue, green, yellow, [orange], red.

Third order: Purple, blue, green, yellow, red, [bluish red].

Fourth order: Green, [yellowish green], red.

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* NEWTON'S 'Opticks,' 3rd Edition, London, 1721, pp. 174 and 206.

In the other orders **Newton's** two lists and our own agree in distinguishing only two colours, green and red.

It was soon found that a more precise method of nomenclature and a convenient system of symbols for the tints were needed. Each colour was therefore sub-divided into ten equal parts which were indicated by numerals from 0 to 9 in the order of increasing thickness. Any tint could be readily expressed by writing down the first letter of its name followed by a bracket containing two numbers, the first of which showed the order, and the second the part of the colour referred to. Thus B [3, 0,] B [3, 9] and B [3, 5] indicate the divisions of the blue of the third order nearest to the purple, nearest to the green, and halfway between the boundaries of those colours respectively. The last of these corresponds to the thickness answering to the same colour in **Newton's** scale.

If a single tint is alone presented to the eye, it is often difficult to assign the precise value to be attached to it. The films, however, generally displayed several colours in the same vertical line, and it was thus possible to reinforce the judgment as to the tint at any point by estimating the distances of that point from the boundaries of the colour in which it lay. The increment in thickness corresponding to the entire range of a colour, varies considerably, and thus the tenth of a tint has also different values in different parts of the scale. As a rule it is less than one per cent. of the thickness, though in the blue of the second order it rises to 3 per cent. It was thus sometimes necessary and possible to estimate to the twentieth of that colour.

A valuable check on the accuracy of the observations was afforded by the use of the two mirrors. The light reflected from the first was incident on the film at about 45°, and the angle of incidence of that reflected from the second was made as small as possible and varied on different occasions between 2° and 7°. When the thickness of the film at any given level was to be measured, the horizontal spider line of a cathetometer-telescope was brought to that level, and the colours were noted at the points were it crossed the two bright strips.

The thicknesses corresponding at normal incidence to the colours at these points (T_1 and T_2), and the true thickness of the film (T) are connected by the equations

$$T=\frac{T_1}{\cos i_1'}=\frac{T_2}{\cos i_2'},$$

where i_1' and i_2' are the angles of incidence on the internal surface of the film. Two simultaneous but independent measures of the thickness were thus obtained, the mean of which was adopted as correct.

This method presented the additional advantage of affording a means of checking the accuracy of the scale of colours employed. The ratio $T_1 \div T_2$ depends only on the nature of the liquid and the magnitude of the angles of incidence. On calculating it for a number of observations, it was soon found that although tolerably constant for any particular colour, it varied considerably for different colours. This shewed that the thicknesses corresponding to the colours were not identical with those furnished by NEWTON'S scale, even when readjusted on account of the omissions already referred to. Nor was this surprising. The above description of the method of estimation is sufficient to show that in all probability no two observers would agree as to the positions of the boundaries between neighbouring tints. If accuracy is required each must construct his own scale.

This might be undertaken by observations on a set of NEWTON'S rings. Two difficulties, however, here arise, the first due to the fact that, even when NEWTON'S rings are produced on a large scale, the colours are far more crowded than on the films, and thus not only is their appearance somewhat changed, but differences of hue clearly distinguishable in the one case are lost in the other. The second difficulty has its origin in the distortion of the lenses, used in the production of NEWTON'S rings, in the neighbourhood of their point of contact. Calculations based on the assumed sphericity of the lenses are thus of little use near the central black patch. For these reasons it was determined to supplement the observations on NEWTON'S rings by those made on the films, to determine the thicknesses corresponding to a few clearly marked colours by the first, and the ratios of the thicknesses of the intermediate tints by the second.

As it was necessary to produce the rings on a large scale, a pair of the curved plates, sold for this purpose by Mr. LADD in a brass case, were placed in a vertical plane. The diameters were measured by the cathetometer. The light employed was reflected from a mirror and passed through a slit about two feet distant from the rings. By placing a soda flame behind the slit, daylight and homogeneous light incident at the same angle could be used in turn. If T be the thickness corresponding to a ring of colour of diameter Δ, which lies between the n^{th} and $(n+1)^{\text{th}}$ dark rings formed by the soda flame, the diameters of which are δ_n and δ_{n+1},

$$T=\frac{588}{2}\left\{n+\frac{\Delta^2-\delta_n{}^2}{\delta_n{}^2{}_{+1}-\delta_n{}^2}\right\}.$$

By thus deducing the value of T from the diameters of the dark "soda" rings in its immediate neighbourhood, errors due to the non-sphericity of the lens are much diminished, and the observation is independent of the closeness of contact between the two plates of glass in the centre of the system.

To obtain accurate results in the neighbourhood of the first dark soda ring it was necessary to determine some additional fixed points. A plate of blue cobalt glass was therefore interposed between the mirror and the rings. The light thus obtained was not homogeneous, but was sufficiently nearly so to make it safe to assume that the thicknesses corresponding to the first few dark rings were proportional to the natural numbers.

The diameters of the four smallest rings were measured, the wave length of the light was computed from the larger pair by means of the above formula, and this being known, the inner pair could be used as reference lines for colours in their immediate neighbourhood. On one occasion, four measures, each made on a single ring, gave 431, 429, 431 and 438 millionths of a millimetre as the wave length respectively; on another occasion two measures gave 432 and 438. As none of these numbers differ from their mean by more than 1·2 per cent. these measurements enable us to estimate the accuracy of the method. Red glass (coloured with oxide of copper) was used in the same way, and two measures agreed in giving the wave length as 615. In the case of the blue of the second order the difficulty was much increased by the fact that the boundary of a colour so near the centre of the rings is very irregular. If we except this tint, however, no measurement ever differed from the mean of the measures on the same colour by more than 1·6 per cent. As three sets of measurements were taken at intervals of nine and six months respectively, and as in most cases three, and in some five, observations were made in each colour, it seems that the errors of recognition and measurement combined did not (except in the case mentioned) exceed 1·6 per cent.

Side by side with these experiments, which were purposely made at long intervals to test the constancy of the colour estimations, other supplementary observations were made upon the films themselves. The scale of colour at first used was obtained by assigning to each tint the thickness ascribed to that of the same name in NEWTON's second scale, and adjusting the boundaries so that the number thus assigned to any tint should be the mean of those assigned to its boundaries. The scale so formed was then tested in the following way. The angles of incidence and the refractive index being known, the ratio of the cosines of the angles of internal incidence could be calculated. The colours at the points where the horizontal cross wire cut the illuminated bands on the film having been noted, the air-thicknesses corresponding to them at normal incidence were taken out from the table of colours, and the ratio of the larger to the smaller calculated and divided by the ratio of the cosines. The number so obtained will be referred to as the *quotient*, and the difference between it and unity was taken as a measure of the accumulated errors of the scale and of the observation. The mean apparent thickness of the film was easily deduced from the quotient.

As has been already stated, the observations proved that the table of colours required revision. This was effected by assigning the quotients to the colours shown by the more obliquely illuminated band in the observations by which they were obtained, and re-arranging the table so as to make the mean of the quotients assigned to each colour as nearly as possible unity. Care was taken to make the revised table agree closely with the observations on NEWTON's rings.

In January, 1879, 600 double colour observations were made on the two illuminated bands, and the table of colours as corrected by them was again revised by means of 1,142 observations made in September and October, 1879. No further change has been found necessary, although the accuracy of the table thus corrected has been further tested by more than 500 observations made in September and October, 1880. It is hardly necessary to state that these revisions were carried out without any reference to the electrical experiments with which the results obtained by them were to be combined. As, however, the conclusions arrived at in this paper with respect to the specific resistance of soap films depend only on the final series of experiments made during September and October, 1880, it is satisfactory to note that the table of colours in no way depends upon these experiments, but that they are used below only to test the results arrived at a year before.

The following details will serve to show the accuracy attained in the two sets of experiments, in the calculation of which the table as finally adjusted was used:—

Experiments made in September and October, 1879.—The films were examined under very varying hygroscopic conditions, and it was thought advisable to see whether the value of the quotient was affected by the varying dilution of the films. The value of the quotient would have been about 2 per cent. greater for pure water than for a standard solution. Twenty films were selected, which gave tolerably constant specific resistances, and the means of the ratios given by observations on each colour were calculated for each film. The mean of these numbers for the ten films which had the lowest specific resistance, and were therefore (as will be shown hereafter) the more dilute, was 1·004, for the other ten films it was 0·994. A precisely similar result was attained when the quotients were compared film by film instead of colour by colour. As the effects of evaporation or absorption seemed, therefore, to appear on the means of a large number of observations, and as we had not at that time all the data for calculating the theoretical quotient in each case, the mean of the quotients for each colour was divided by the mean of all the quotients. These numbers are inserted in column VII. of Table II.

Experiments made in September and October, 1880.—These experiments being, for the reasons already stated, the more valuable as tests of the table of colours, we give the following rather fuller details:—516 observations were made, and (omitting fractions) in the case of 52, 84, and 95 per cent. of these the difference between the quotient and unity was less than ·01, ·02, and ·03 respectively. Hence in 84 per cent. of the observations the mean value adopted differed from both of those from which it was derived by less than 1 per cent. The mean value of the mean quotients for all the colours was 0·9985. The ratio of the two cosines was the same in all cases, viz.: 1·157. Both these numbers were calculated with the refractive index of the standard solution.

In comparing the new table with that of NEWTON, it is best to take from the latter only such colours in the second list as have on each side of them colours which also occur in our own list, *i.e.*, to confine the comparison to those cases when no question as to the boundaries of the colours can arise.

In Table II.—Column I. contains the names of the colours.

Column II. the symbols.

Column III. the thickness in terms of 10^{-5} centims., according to NEWTON'S scale.

Column IV. the thickness according to the observations on NEWTON'S rings.

Column V. the thickness according to the corrected scale.

Column VI. the difference of thickness corresponding to a tenth of a tint in each of the colours measured.

Column VII. the mean quotient for each colour obtained in September and October, 1879, each (for the reasons given) divided by their mean.

Column VIII. the mean quotients obtained in September and October, 1880, also divided by their mean to make them comparable with the preceding.

Column IX. the difference between unity and the mean of the numbers in columns VII. and VIII., expressed in terms of 0·001.

TABLE II.

I.	II	III.	IV.	V.	VI.	VII.	VIII.	IX.
First order—								
Red	R (1, 5)	. .	. .	2·84	·032	. .	1·003	+ 3
Second order—								
Violet	V (2, 5)	. .	.	3·05	·011	. .	1·010	+10
Blue	B (2, 5)	. .	3·34	3·53	·081	1·000	0·998	− 1
Green	G (2, 5)	3·78	. .	4·09	·034	1·003	0·999	+ 1
Yellow	Y (2, 5)	4·07	4·50	4·54	·054	0·998	1·000	− 1
Orange	O (2, 5)	4·31	. .	4·91	·022	0·996	0·995	− 4
Red	R (2, 5)	. .	. .	5·22	·040	1·006	1·003	+ 5
Third order—								
Purple . . .	P (3, 5)	. .	. .	5·59	·035	0·996	0·995	− 4
Blue	B (3, 0)	. .	5·73	5·77	·053	0·997	0·993	− 5
„	B (3, 5)	5·85	. .	6·03	. .	. .	. .	. .
Green . .	G (3, 5)	6·29	. .	6·56	·053	1·006	1·000	+ 3
Yellow . . .	Y (3, 5)	6·79	7·11	7·10	·056	1·001	1·002	+ 2
Red . . .	R (3, 5)	7·25	. .	7·65	·051	1·003	0·994	− 1
Bluish-red . . .	BR (3, 5)	8·00	. .	8·15	051	1·001	0·997	− 1
Fourth order—								
Green . .	G (4, 0)	.	8·58	8·41	·103	0·997	1·004	+ 1
„	G (4, 5)	. .	. .	8·93	. .	. .	. .	. .
Yellow-green . .	YG (4, 5)	9·00	9·71	9·64	·041	1·004	1·002	+ 3
Red	R (4, 5)	10·08	. .	10·52	·134	0·994	0·997	− 4
Fifth order—								
Green	G (5, 0)	. .	11·29	11·19	·141	0·995	1·000	− 2
„ . . .	G (5, 5)	11·50	. .	11·88	. .	. .	. .	. .
Red	R (5, 0)	. .	12·66	12·60	·150	0·996	0·997	− 3
„	R (5, 5)	13·12	. .	13·35	. .	. .	. .	. .
Sixth order—								
Green	G (6, 0)	. .	14·01	14·10	·138	. .	1·001	+ 1
„	G (6, 5)	14·68	. .	14·79	. .	. .	. .	. .
Red	R (6, 0)	. .	15·44	15·48	·157	. .	1·009	+ 9
„	R (6, 5)	16·25	. .	16·27	. .	. .	. .	. .
Seventh order—								
Green	G (7, 0)	. .	16·99	17·05	·164	. .	. .	. .
„	G (7, 5)	17·75	. .	17·87	. .	. .	. .	. .
Red	R (7, 0)	. .	18·49	18·69	·135	. .	. .	. .
„	R (7, 5)	19·25	. .	19·36	. .	. .	. .	. .
Eighth order—								
Green	G (8, 0)	. .	20·04	20·04	·111	. .	. .	. .
Red	R (8, 0)	. .	21·15	21·15	. .	. .	. .	. .

We are unable to place much confidence in the observations of the violet of the second and the red of the first order, as they were very few in number. Even in the case of the blue of the second order, though the quotients agreed well together, the difference between the figures in columns IV. and V. is too large to be satisfactory. We have therefore in most of our work taken 4×10^{-5} centims. as the smallest apparent thickness which could be used where great accuracy was required. For the rest, the difference between the observations on NEWTON's rings and the corrected table rarely exceed one per cent., while NEWTON's scale in parts differs from both by as much as ten per cent. of the thickness. On the whole we think the optical observations alone show that the corrected scale above the blue of the second order is accurate to one per cent.

V. *Method of Experiment.*

In our previous experiments, described in the paper already referred to, a cylindrical film was formed, supported above and below by cylindrical platinum cups, and its resistance was measured by WHEATSTONE's bridge. This method was unsatisfactory, first, because it took no account of any "polarisation" which might occur at the extremities of the film; and secondly, and this is the more important disadvantage, because it compelled us to measure the resistance of the film as a whole, instead of selecting any part which might be suitable to our purpose.

The method since adopted and now to be described is free from the above-mentioned disadvantages. It depends upon the measurement of the difference of potential, produced by the passage of a current, between two horizontal sections of the film. Steel sewing needles or gold wires (preferably the latter) are inserted in the film, and the difference of potential between them measured by an electrometer, and compared with that between two other points in the same circuit separated by a known resistance. Three needles were actually employed, supported horizontally, and about 15 millims. apart, the upper one being about 5 millims. below the upper cylindrical cup, and the lowest 10 millims. from the lower cup. Calling the needles 1, 2 and 3, counting from top to bottom, the difference of potential between 1 and 2, between 2 and 3, or finally between 1 and 3 could be measured as might be desirable. It is clear that this method disposes of any difficulties that may arise (1) from counter-electromotive force set up at the electrodes by the passage of the current, and (2) from irregularities in the substance of the film itself at or near its supports. Observation shows that such irregularities are of frequent occurrence. For example, it happens not unfrequently that a film, after thinning continuously for some time, begins to thicken from the bottom, the colours rising and being so crowded together that it is difficult to distinguish them. Again, white flecks occasionally appear at the top, and more frequently black patches, forming themselves gradually into a ring of black, the resistance of which is, as we have already shown,* enormously great in comparison with that of a coloured ring of the same width, even when the colour is of the second or first order.

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* *Loc. cit.*

In discussing the method it is necessary to show that no sensible error was caused by the disturbance of the equipotential lines which accompanied the introduction of the needle points into the films. The distance between any two needles being greater than those between the upper and lower needles and the metal rings by which the cylinder was supported, it will be sufficient to consider the extremities of the film only.

A small aggregation of liquid was always observed around the needle points. Its dimensions were variable, but the diameter was never greater than a millimetre and a half. If we assume that the resistance of this mass (of radius a) was infinitely small, compared with that of the surrounding film, and that the dimensions of the cylinder were infinitely greater than a, the equation to each of the disturbed equipotential lines would be of the form

$$y\left\{1-\frac{a^2}{x^2+y^2}\right\}=\text{constant},$$

where the axes of x and y are horizontal and vertical respectively.

The maximum displacement, by the introduction of a needle, of the equipotential line which passes through $x=0$, $y=\eta$ would therefore be $\frac{a^2}{\eta}$. The condition of accuracy is that this quantity shall be negligible at a distance from the needle equal to that of the metal ring which bounds the cylinder. This distance was generally greater than 5 millims., and therefore 0·11 is the limit of error. The distance between two needles being 15 millims., it is therefore possible that an occasional error of 0·75 per cent. may have been caused, but we doubt whether it ever reached this amount; and in all the more important experiments, viz.: those made after September 20, 1880, it certainly did not, as the two lower needles were exclusively used, and the error must therefore have been halved.

The thickness of the film between two needles was taken as the mean of the thicknesses at the needles. These were measured in turn, the time of each observation being noted, and curves were drawn of which the ordinates represented the thicknesses and the abscissæ the times. It was found that to make and record a measure of the thickness occupied about half a minute, but, when the film was thinning slowly, observations were made at longer intervals. The curves were used to correct errors of observation by smoothing off irregularities. On a few occasions errors amounting to 2 per cent. of the thickness were thus detected. In the curves themselves the error of observation probably nowhere exceeds 1 per cent., and approaches that amount only when the films were very thin.

The electrical observations on the resistance of the film were carried on simultaneously with the above, and curves showing the relation between the resistance and time were in like manner constructed. By taking out from these two sets of curves the thickness and resistance at any particular time, the specific resistance could be calculated by the formula given in our previous paper.

When the films became thin they were generally so uniform in tint that it was sufficient to determine the thickness half way between the two needles.

VI. *Description of the Apparatus.*

The experiments were made in a room with a south aspect, the film apparatus being supported in the window recess. Plate 59, fig. 1, represents a sectional elevation of this apparatus from front to back, taken very nearly through the centre. Plate 60, fig. 5, shows the same in plan, with the mirrors and the telescope of the cathetometer. A and B (fig. 1) are iron cylindrical cups, each 33 millims. in diameter, placed vertically one above the other, between the edges of which the cylindrical liquid film is formed. A small brass tube C, attached to A, slides with easy friction through another tube F, and enables the length of the film to be adjusted at will. F has a brass flange which carries a binding screw, and is cemented to a glass disc G, for closing the circular aperture at the top of the glass case HH. Through this aperture the cup A, with attached tube, &c., can be removed from the case for purposes of cleaning. A piece of indiarubber tubing fastened to the end of the tube C passes through the base board of the apparatus at the point *a* (fig. 5), and is connected with a reservoir of air.

The lower cup B is soldered to the top of a brass tube DD, which slides in a slightly larger tube, and can hence be removed when required. The smaller brass tube E was originally intended to afford a means of filling or emptying the cup from the outside. In the later experiments it was not used for this purpose, but, its upper end being stopped up with sealing wax, it merely served as a metallic connexion between the film and the source of electricity.

A film, when under examination, is surrounded by two glass cases, the front and side faces of which are constructed of specially polished glass. The top of each is of ordinary plate glass, and the back of wood. The inner case HH fits tightly down by its wooden base upon the square ebonite platform I, by which the lower cup is insulated. The outer case H′H′ slides down upon the inner by grooves *cc*, fig. 5, and is held firmly in its place by the back of the inner case, and by the base board I′ (fig. 1). The object of the outer case is to prevent sudden changes of temperature or hygrometric state in the interior of the apparatus.

The needles (*nnn*, fig. 1, shewn also in figs. 2, 3, and 5), were, in the experiments described in this paper, made of gold wire. They pass through small holes drilled in a vertical pillar of ebonite. Fine silk-covered copper wires soldered to their extremities make connexion with the mercury cups M, and hence with the electrometer. To provide against the ebonite pillar being wetted by the bursting of the films, and the insulation of the needles from each other, and from the base of the apparatus being thereby impaired, a small glass shield is also supported by the pillar (figs. 1, 2, 3), in which holes are drilled large enough for the needles to pass through without touching it.

Figs. 2 and 3 show in detail the arrangements for moving the needles from the outside when the inner and outer cases are in their places, and for adjusting them to their proper position in the film. A brass tube, *g*, passes through the chamber below the glass case, and has at its lower end a milled head, *h*, by which it can be rotated about its axis. A brass rod, *f*, passes through it, having below a milled head, *k*, and above a pinion, *e*, which engages with the rack, *d*, carrying at one extremity the ebonite pillar and gold needles. It will thus be seen that motions of two distinct kinds can be given to the needles from without; they may be moved to and fro parallel to themselves by turning the milled head, *k*, which works independently of the outer tube, and they may be turned round a vertical axis by means of the milled head, *h*. In the figures the needles are represented at right angles to the rack, *d*, but they may be placed at any other suitable angle by turning the ebonite pillar in its socket.

Arrangement for forming the films.—This is shown on the right hand side of figs. 1 and 5. A rectangular portion of the ebonite base board is cut away, and communication thus made between the interior of the glass case and the chamber PP beneath. This aperture can be closed by a wooden trap-door, shown in dotted lines in fig. 1, and turned up or down by a milled head, *t*, at the back of the case. When it is turned down, its edges dip into a rectangular groove filled with mercury. K is a brass rod with screw thread cut upon it, which serves as a rack for the pinion, worked by the screw head, *l*. It carries eccentrically at the top a circular brass platform, on which is placed a shallow glass dish, of diameter a little greater than that of the cylinder A, containing the liquid for forming the films. The position of the rod K is such that when the platform is raised through the aperture by the screw head, *l*, and rotated by the screw head, *m*, the glass dish can be brought immediately under the cylinder A, and a horizontal film thus formed over the mouth of the latter. This done, the dish is lowered through the aperture which is then closed by the trap-door.

The object of the lower chamber PP will now be evident. It serves as a storehouse for the liquid, which can be thus introduced into and withdrawn from the film-chamber without establishing any connexion between the latter and the outward air. Its base is of ebonite, with holes in it for the electrometer wires to pass through. The cylinder DD is in contact with no material except ebonite.

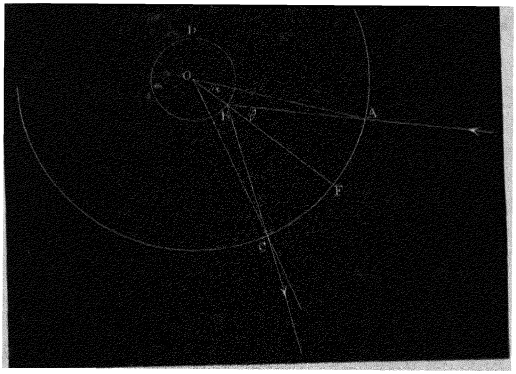
The horizontal film mentioned above is converted into a cylinder as follows:—Air being forced from the reservoir, the film becomes spherical, and increases in size until it adheres to the edge of the lower cup B, which is about three quarters full of the same liquid. It is then only necessary to withdraw air until the barrel-shaped film becomes cylindrical, an adjustment which can be accurately made with the telescope.

The needles are now made to pierce the film, an operation which can in general be effected without endangering its existence, and their position so adjusted that the points at which they meet the film may lie upon one of the vertical bands of light seen from the cathetometer.

Method of measuring the angle of incidence.—In Plates 59, 60, figs. 1 and 5, NN is a stout mahogany board which supports the whole apparatus. It has the form of an incomplete circle, the centre of which coincides with the centre of the cup B, and a paper scale graduated to half degrees is attached to its outer edge. It rests upon a long wooden slab SS, fixed in the window recess, by three points, viz.: the knob *q*, and the two levelling screws *p*, *p*. A portion of the slab SS has been cut away to make room for the lower chamber.

T is the telescope of the cathetometer, placed at a distance of about 2 metres from the film, and Q and R are the mirrors for reflecting light upon the film. The angle of incidence in the case of either mirror was determined in the following manner:—

In the accompanying figure, let B D represent the film and A F C the graduated circle, and let the light fall on the film in the direction A B and be reflected along B C. Join O A, O C.



If

$$OA(=30{\cdot}4 \text{ centims.})=a, \text{ the angle } AOB=\alpha$$
$$OB(=1{\cdot}65 \text{ centim.})=b, \quad ,, \quad ,, \quad ABF=\theta$$

then

$$\frac{a}{b}=\frac{\sin\theta}{\sin(\theta-\alpha)},$$

whence

$$\tan\theta=\frac{a\sin\alpha}{a\cos\alpha-b}$$

and thus θ is known if α is known.

The vertical cross wire of the telescope of the cathetometer being directed to the centre of the illuminated band, a plumb line is passed along the edge of the graduated circle, until it arrives at the point C, when it coincides with the cross wire. This point is noted on the circle. Similarly when the plumb line is at A, its image formed by reflection coincides with the cross wire. We thus know the arc AC and consequently the angles α and θ.

Successive readings made in this way rarely differed by more than 10′, and as the mean of several determinations was always taken, the accuracy of the method was amply sufficient for our purpose. The values of α during the experiments described in this paper were 42° 41′ and 2° 44′.

The Hygrometer.—Fig. 4 represents the hair hygrometer and the thermometer attached to it. The hair was cleaned according to the method prescribed by REGNAULT. The index is of ivory, and is graduated in degrees of arc. When the instrument was placed under a receiver, dried by sulphuric acid, and left for 24 hours, the index stood at 0°. When the receiver was saturated with aqueous vapour the index stood at about 60°. In the experiments described, the hygrometer was only used for the purpose of ascertaining whether any change in the hygrometric state of the air surrounding the film took place while the film lasted. No attempt was made to obtain the actual values of the hygrometric state corresponding to the indications of the instrument.

The position of the hygrometer in the case is shown at V, fig. 5. A strip of glass was placed in front of it to protect it from the spray caused by the bursting of the films.

Method of maintaining a constant hygrometric state.—In many of the experiments it was necessary to prevent as far as possible any absorption of vapour by or evaporation from the films. To check evaporation, the lower cup B was filled with liquid to within about 2 millims. of the edge. Two dishes containing the liquid and exposing an evaporating surface of about 20 square centims. were also placed in the inner chamber. On and after September 24 these were replaced by a much larger dish with an evaporating surface of about 52 square centims. Sheets of blotting paper moistened with the liquid were attached to the back and one of the sides of the inner case. One of these dipped in a small vessel containing the liquid and could be moistened by a pipette introduced through a small hole cut in the back of the apparatus.

As the inner glass case, though fitting the ebonite base board, I, sufficiently well to prevent the ready passage of air or aqueous vapour, was not absolutely air tight, small dishes containing the liquid with pieces of blotting paper, also moistened with the liquid, dipping into them, were placed in the outer case. A dish of the liquid was also placed in the chamber below the case.

The air for forming the films was contained in a bladder, fitted with pressure boards. On being forced from the bladder it passed first through two tubes containing caustic potash and pumice soaked in strong sulphuric acid respectively, and then over a surface of *liquide glycérique* contained in a horizontal tube about 2 centims. in diameter. When a constant state was required, the first film made was never used; it was left until the hygrometer and thermometer had become steady, was then broken without opening the case by giving it a sharp blow with the needles, and replaced by another. If during this operation the hygrometric reading altered by more than one or two tenths of a division, the second film was also rejected, and so on. In general, the films used had been formed without any appreciable change in the hygrometric state. Minute and even excessive as these precautions may appear, we found, as will be shown hereafter, that they were barely sufficient for our purpose.

The principal use of the chamber P was to enable us to keep the liquid used in making the films unchanged, when for special purposes we wished to charge the inner case with aqueous vapour at a greater or less tension than that proper to the *liquide glycérique.*

Insulation tests.—As before stated, the ebonite pillar carrying the three needles was protected from the spray of the bursting films by a glass shield, with holes in it for the needles to pass through. As, however, the precautions just described filled the case with large quantities of vapour, it was possible that in the course of a few hours a deposit of moisture might be formed on the ebonite pillar, or upon the glass sides of the apparatus, of such conducting power as seriously to interfere with the accuracy of the experiments. To guard against error from this source, the insulation of the needles from each other, and of the needles and the upper and lower cups from the case, was tested at the end of each set of observations. The observations were regarded as satisfactory so long as the greatest resistance measured did not exceed 0·3 per cent. of the insulation resistance. When, as happened on one or two occasions, it greatly exceeded this limit, the preceding observations were rejected.

VII.—*Description of method employed for measuring the electrical resistance of the films.*

The battery consisted of 4 LECLANCHÉ cells of the ordinary pattern. A simple circuit was formed, comprising the battery, three boxes of resistance coils, and the film to be experimented upon. The resistances consisted of (1) a box of coils ranging from 1 to 10,000 ohms, (2) ten coils each of 10,000 ohms, and (3) ten coils each of 100,000 ohms. In the case of most of the films recently observed, the resistance in the boxes was either 500,000 or 1,000,000 ohms, and two binding screws, one on each side of this resistance, could be connected by carefully insulated wires with the electrometer. The current always passed through the film from the top to the bottom.

The Electrometer.—The electrometer used is one of Sir WILLIAM THOMSON'S quadrant instruments, made by WHITE; it retains its charge remarkably well, a few turns of the replenisher being sufficient to bring the hair up midway between the two dots after the instrument has been left to itself four or five days, and no re-charging has been found to be necessary during the last eighteen months. We have found it convenient to connect the outer case of the electrometer to earth, and one of the electrodes to the induction plate. This, though not the most sensitive arrangement, has been found to give the best results. The two electrodes are also connected by fine silk-covered wires to short copper rods which dip into mercury cups placed on a sheet of glass immediately in front of the electrometer. Two other mercury cups, connected with the points whose difference of potential is required, stand close by, and gutta-percha-covered copper wire bridges, which can be arranged parallel or cross-wise, enable the electrification of the quadrants to be reversed, and readings to be obtained right and left. This arrangement was found to work quite satisfactorily, and was more to be depended on than Sir WILLIAM THOMSON's reversing key, which was tried for a time and abandoned.

The deflection of the needle of the electrometer, as measured on the scale, not being proportional to the difference of potentials of the quadrants, it was necessary to correct it. It was thought that, A—B being the difference of potential of the quadrants and d the deflection, a value might be assigned to μ which would make A—B approximately proportional to $d\left(1-\frac{d}{\mu}\right)$; and it was found on trial that the value $\mu = 10{,}000$ satisfied the requirements sufficiently well.

A circuit was formed containing a battery of six DANIELL's cells and a constant external resistance of 12,000 ohms. Two points in the circuit separated by a resistance varying from 1,000 to 6,000 ohms were connected with the electrometer, and the deflections read off. The following table shows the proportionality between the deflections corrected according to the above rule, and the difference of potential obtained by multiplying the smallest corrected deflection by 2, 3, 4, 5, 6 in succession. The agreement may be regarded as perfectly satisfactory, when it is remembered that the electrometer readings cannot be relied on to much less than 0·5 of a division.

TABLE III.—Calibration of electrometer scale.

Resistance $=1000\,n$.	Deflection ($=d$).			$\frac{d^2}{10^4}$	$d-\frac{d^2}{10^4}$	$52\,n$.
	Right.	Left.	Mean.			
6000	328·5	320	324·2	10·5	313·7	312
5000	266·5	267	266·7	7·1	259·6	260
4000	214·5	210	212·2	4·5	207·7	208
3000	158·5	159	158·7	2·5	156·2	156
2000	104·5	106·5	105·5	1·1	104·4	104
1000	52·5	52	52·2	0·2	52·0	52

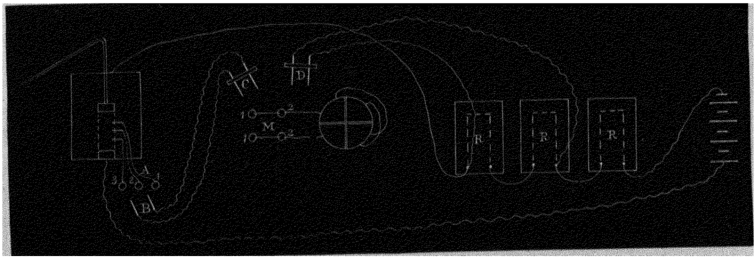
The distance of the electrometer from the scale and lamp was about a metre, and the deflections were, as a rule, within the limits 80 and 250 divisions. In some of the experiments the lamp and scale were removed to a distance of 1·6 metres from the electrometer, and a sharp image of the slit on the scale obtained by means of a lens. In both positions of the scale it was found that the above empirical rule was approximately true, and that the difference of potential corresponding to a given electrometer deflection d was proportional to $d\,(1-d \times 0{\cdot}0001)$. Similar observations were from time to time repeated, and always with the same result.

According to MAXWELL ('Electricity and Magnetism,' vol. i., p. 274), the deflections of the electrometer, when of moderate magnitude, are proportional to the product

$$(\mathrm{A}-\mathrm{B})\left(\mathrm{C}-\frac{\mathrm{A}+\mathrm{B}}{2}\right)$$

where C is the potential of the needle and A and B those of the quadrants. Now, in calibrating the electrometer, the potential of one pair of quadrants (say B) was kept constant, and the other gradually raised, and thus our empirical correction is made up partly of the conversion of arcs into tangents, and partly of the change due to increasing values of A+B. But since a correction which holds for one constant value of B will not hold for another, it was necessary to test the validity of our formula for the range of potentials studied. For this purpose two resistances, each of 100,000 ohms, were included in a circuit, and a piece of carbon paper with a resistance of about 700,000 ohms interposed between them. The mean electrometer deflections corresponding to the equal resistances were 127·85 and 127·55, which differed by only 0·3 of a division. Hence the formula may be regarded as valid, whatever be the position in the circuit of the resistance considered.

Fig. 6.



Method of observation.—The order of observations of the electrical condition of a film was as follows: The film having been formed and adjusted, and the needles fixed in their proper places, the copper wires B (fig. 6) were inserted in the mercury cups belonging to the particular pair of needles between which the resistance was to be measured. The connecting pieces C and D, consisting each of a strip of ebonite with stout copper wires passing through, were inserted in succession in the mercury cups (M 1, 1), and the deflections written down in order, the time of each observation, given by a chronometer, being noted by the side.

It was generally found that the needles were of slightly different potentials even when no current was passing, and it was necessary to correct for this which would affect the apparent resistance of the film. This difference was measured after the film was formed and before the current was allowed to traverse it. It was generally small, and sometimes vanished, but as a rule caused a deflection of from 1 to 5 divisions of the scale. Frequently its direction was constant for the same film, and even for the same day, but, as its magnitude could not be relied on to remain unaltered during the life of a film, the current was interrupted at intervals of 5 or 6 minutes and the condition of the needles examined. The deflections observed were added to or subtracted from those obtained for the needles when the current was passing, according to their direction.

The same phenomena were observed with steel and platinum needles in an augmented degree, and were less under control. It was always found that the difference of potential between needles 1 and 3 was equal to the sum of the differences between 1 and 2, and 2 and 3. This difference was not altered in any constant manner by the passage of the current. Sometimes during the life of a film and with the current continuously passing it increased, sometimes it diminished, and again at other times showed no signs of alteration. But in what direction it would change, if at all, could not be predicted.

Polarisation so-called has nothing to do with the phenomenon, since its direction and magnitude do not depend upon the direction or continuance of the current. The cause is to be found in the difference of chemical condition of the needles produced by the action of the liquid quite independently of the current. The needles are affected to a different extent, and thus a difference of potential is set up between them. That this is the cause of the phenomenon was shown as follows: Two steel needles, not specially cleaned, were inserted in a film, and their extremities connected by wires with the electrometer. The deflection was +6. The lower needle was then passed through the fingers, and the deflection rose to +9. The lower needle was then washed, rubbed with emery paper, washed again, and connected with the electrometer, which now showed a deflection of −3. Thus by cleaning the needles unequally, either could be made to be of a higher potential than the other. With gold wires these changes occurred to a less degree than with either steel or platinum, and hence gold was the metal employed in the latest experiments. In these it was usual each morning to wash the gold needles in boiling nitric acid and then in distilled water. But even when the greatest care was taken in cleaning the needles, it was impossible to ensure their being in such a state as to exhibit when immersed in the film no difference of potential.

As a specimen of the observations made in measuring the resistance of a film, we give the following: The symbols × and ∥ mean that the wire bridges of the electrometer reversing arrangement are crossed and parallel respectively.

SEPTEMBER 27, 1880.—Film I., needles 2–3, resistance in boxes = 508,000 ohms = R.

Part of circuit which is connected with electrometer.	× Deflection.	Time.			∥ Deflection.	Time.		
		h.	m.	s.		h.	m.	s.
Needles (no current passing)	0	..			2 right	Mean = 1		
Film	80 right	12	54	12	75·5 left	12	55	6
Resistance	174		54	42	166		55	30
Film	82·5		56	0	79		57	0
Resistance	167		56	30	159·5		57	30
Film	85·5		58	0	82		59	0
Resistance	159·5		58	30	151·5		59	30
Needles	0	..			2 right	Mean = 1		
Film	91·5	1	1	0	85·5	1	2	0
Resistance	149·5		1	30	140		3	12
Film	93		3	42	90		4	30
Resistance	140		4	6	133·5		5	0
Film	96·5		5	30	91·5		6	30
Resistance	135·5		6	0	127		7	0
Needles	·5 right	..			2·5 right	Mean = 1		

These observations were continued until $1^h\ 22^m$, when the film burst. The numbers thus obtained were then plotted down on a sheet of curve paper, times being represented as abscissæ and deflections as ordinates. Thus, four curves were drawn, of which F (right) and F (left) gave the mean film curve, and R (right) and R (left) the mean resistance curve. The curves F (right) and F (left) were always very close together and sometimes coincident, but, as the zero of the electrometer was liable to change, the curves for both right and left deflections were always drawn and the mean taken. The same remarks apply to the curves of resistance-deflections. From the two mean curves thus obtained a third is deduced as follows: The ordinates of the curves are read off at points separated by intervals of four minutes, and when the proper corrections, described above, have been made, the film-deflection is divided by the resistance-deflection and the quotient multiplied by the resistance of the coils. A curve passing through the series of points so obtained enables us to determine the resistance of the film between the two needles at any epoch.

Example.—September 27. Film I., at 12^h 58^m the deflections are, for film 83 [or 84 when allowance is made for the independent difference of potential of the needles], and for resistance 159·6.

$$84 \; (1 - 84 \times 10^{-4}) = 83{\cdot}4 \text{ nearly}$$
$$159{\cdot}6 \; (1 - 159{\cdot}6 \times 10^{-4}) = 157 \quad \text{,,}$$

Hence the resistance is $\frac{83{\cdot}4}{157} \times 508{,}000 = 269{,}400$ ohms.

VIII. *Method of testing the accuracy of the experiments.*

Observations with the Galvanometer.—With the object of ascertaining whether the apparatus was in satisfactory working order, the following experiment was occasionally made. A strip of Dr. MUIRHEAD's carbon paper was attached to a piece of glass by brass clips and carefully insulated. Its extremities were connected to a WHEATSTONE's bridge and also to the electrometer, and its resistance measured by the two methods in succession. Usually an observation by galvanometer was made between two observations by electrometer, care being taken to prevent changes of temperature of the carbon paper. It is unnecessary to give details of these experiments, but the following may be quoted as indicating the character of the results obtained :—

	Galvanometer.	Electrometer.	Difference.
			Per cent.
September 24 . .	1,620,000	1,633,000	0·8
,, 25 . .	1,455,000	1,457,000	0·14
,, 26 . .	1,910,000	1,920,000	0·52

The differences in the resistance of the carbon paper from day to day were due partly to differences of temperature, but chiefly to changes in the hygrometric state of the atmosphere.

The experiments with the carbon paper were repeated at intervals. If the difference of the resistances obtained exceeded 1 per cent. the observations on the films were interrupted, until, the apparatus having been overhauled, the two methods gave results with the carbon paper in sufficiently close agreement. A similar experiment was sometimes performed with the films themselves. The specific resistance was calculated both from measurements made on a portion of the film by the electrometer, and also from the resistance of the whole length measured by WHEATSTONE's bridge. To effect this, the thicknesses at top and bottom, and, if necessary, at intermediate points, were measured.

These observations confirmed our opinion of the superiority of the electrometer method. Results were sometimes obtained agreeing to about 0·5 per cent., but generally the difference was greater, and sometimes it was as large as 6 per cent. The fact that the most discordant results were obtained with films which had exhibited a considerable number of black specks at the top shows the advantage of working in the middle of the film. We may here remark that the effect of these black specks upon the electrometer was carefully noted. They generally travelled round the film at varying speeds, and it was found that a speck much larger than was usual was required to produce any effect on the electrometer by its passage over the nearest needle.

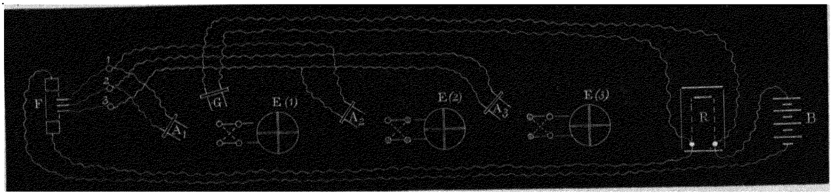
Observations with three Electrometers.—It may be desirable to state why three needles were used for the films rather than two. The object in view was twofold. In the first place, when a film begins to thin, it often happens that its upper part, the region between needles 1 and 2, shows recognizable colours, and is in a fit condition for observation and measurement, before the colours of the region between needles 2 and 3 become well defined, and thus time is gained by attacking the upper part of the film while the lower part still remains thick. Again, it happened not unfrequently that during the observation of the region between needles 1 and 3, a thickening of the film would occur from below, the colours rising steadily from the lower cup and being separated by a sharp line of (apparent) discontinuity from the portion—often perfectly uniform in tint—towards which they were advancing. Such a "thickening from below" was always carefully watched, and as soon as it approached needle 3 the electrical connexions were readjusted and the examination transferred to the region between needles 1 and 2.

In the second place, it appeared very important to be able to prove by direct experiment that the electrometer method of measuring the conductivity of a film was to be relied on, and that the results obtained were not dependent upon the conditions of the instrument employed. The galvanometer method of testing the electrometer measures was not, as has already been stated, always applicable. It was therefore determined to measure the resistances of the film between needles 1–2, 2–3, and 1–3 respectively by three independent instruments. We were fortunate in securing, through the kindness of Professors W. G. ADAMS and F. GUTHRIE, the loan of two quadrant electrometers similar to the one already in use. The electrometers were arranged as shown in fig. 7.

The first operation was to compare the electrometers with each other, which was effected in the following manner:—A simple circuit was formed, traversed by a current of constant strength, the resistance in it being about 15,000 ohms. Two points in the circuit, separated by resistances gradually increasing from 1,000 to 12,000 ohms, were connected in succession with the electrometers, and the deflections, right and left, noted. The numbers so obtained enabled comparison curves to be drawn, by which deflections of electrometers (2) and (3) could be referred to scale readings of electrometer (1).

The comparison of the electrometers was repeated each day while the experiments were going on. When a film was to be examined the connecting pieces, A_1, A_2, A_3, being all inserted in the corresponding mercury cups, the deflections right and left of the three electrometers were observed, the time of each reading being noted. A_1 was then removed, and G inserted in its place, and the deflections right and left again observed. These observations were repeated without cessation so long as the film lasted. The time occupied in taking a complete set of readings for the three needles and for the resistance was from 2 to 3 minutes.

Fig. 7.



F is the film with the three needles inserted in it, connected respectively to the mercury cups 1, 2, 3.
E (1) the electrometer which measures the difference of potentials between needles 1 and 2.
E (2) ,, ,, ,, ,, 1 and 3.
E (3) ,, ,, ,, ,, 2 and 3.
R, box of resistance coils.
B, battery of 4 Grove's cells.

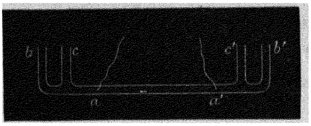
The deflections being plotted down on curve paper, as before described, curves representing the actual resistances between needles 1–2, 2–3, and 1–3 were deduced. The resistance between needles 1 3 ought of course to be equal to the sum of the resistances between needles 1–2 and 2–3. Sometimes the agreement was perfect, and the difference was rarely in excess of 1 per cent.

IX. *Description of the Apparatus for measuring the resistance of the liquid in mass.*

The method finally adopted for measuring the electrical resistance of the soap solutions was in principle the same as that employed in the examination of the films. A current was passed through a known resistance and a tube containing the liquid to be examined; and the difference of potential between two points in the tube which could be connected with an electrometer was compared with that between two other points in the same circuit separated by a known resistance.

A piece of tube was selected, as nearly cylindrical as possible, about ·8 centim. in diameter and 32 centims. in length. Its two ends were bent at right angles to it, as shown in the accompanying diagram, and two short pieces of the same tubing were fused into it near the turned up ends for the insertion of thermometers. At *a*, *a′*, about

13 centims. apart, two fine holes were drilled in the tube, and pieces of platinum wire, the ends of which just projected beyond the inner surface of the glass, were cemented into these by sealing wax. The distance between these two points was measured by



a cathetometer and the diameter was determined by weighing the quantity of mercury which occupied a portion of the tube a little longer than the distance between a and a'.

If

l = the distance between a and a';
r = the radius of the tube;
R = the resistance in ohms of the liquid between the two points;
ρ = the specific resistance, *i.e.*, the resistance per cubic centimetre;

then

$$\rho = R \times \frac{\pi r^2}{l}$$

The value of the constant $\frac{\pi r^2}{l}$ for the tube with which all the soap solutions were measured was ·030001. The specific resistances were thus readily obtained by multiplying the observed resistances by ·03.

The points a, a' were sufficiently far from the bent up portions of the tube to ensure that the lines of flow between them would be parallel to the generating lines of the cylinder, and, as in the films, the platinum points a, a' would not sensibly interfere with this symmetrical flow.

The two electrodes connecting with the battery consisted of flat spirals of platinum wire, the diameters of which were such as to allow them to move freely in the tubes b, b'. The object of using a spiral wire rather than a continuous disc was to facilitate the escape of the bubbles of gas formed by the decomposition of the liquid.

The thermometers used had a range of 40° C, and were graduated to 0°·5, but could be read to 0°·1 without difficulty. They were compared with a thermometer graduated to 0°·1 which had been verified at Kew. Pieces of cork through which they passed supported them in their places in the tubes c, c'.

It is well known that the electrical resistance of a liquid depends in a very important degree upon its temperature; and hence the necessity of knowing accurately the temperature of the soap solution at the time when its resistance was measured. To this end a water bath was contrived containing a chamber sufficiently large to hold the resistance tube, thermometers, &c.

The water bath, which was made of tin plate, is represented in Plate 60 fig. 8, with the front detached for the purpose of showing the paddles, B, B, B, B, for stirring the water and producing uniformity of temperature throughout. These are moved up and down by means of strings passing over a pulley. The rectangular chamber A, A, A, A, is 33 centims. long, 21 centims. high, and 20 centims. deep. It is surrounded on four sides by water, and is closed on the remaining sides, viz.: front and back, by glass windows. In these experiments the window at the back was not used as such, being covered by thick felt. The window in front is provided with felt packing, and is held firmly in its place by four nuts. The bath is supported on an iron stand about a foot from the table.

Four tubes C, C, D, D, each about 2·5 centims. in diameter, pass through the base of the apparatus and are closed by caoutchouc stoppers. Through C, C pass the battery wires, and through D, D two other wires connecting with the electrometer. Similarly three tubes E, E, E pass through the upper part of the apparatus, the standard thermometer F being supported in the centre one. H, H are mercury cups for connecting the electrodes of the tube with the battery wires; and K, K similiar cups for making the electrometer connexions. By placing BUNSEN burners beneath the lateral bent tubes G, G′, which are also made of tin plate, the water could be heated. As a rule the range of temperature employed was between 10° and 35°; within these limits it was easy to obtain any required temperature and to maintain it constant during an interval sufficiently long for two or three complete sets of observations to be taken. No observations were made until the thermometers *f*, *f* gave the same readings and the thermometer F indicated a temperature not differing by more than two or three tenths of a degree from that of the others. The mean between the readings of *f*, *f* and of F was then assumed to be the temperature of the liquid in the tube.

The mode of connecting with the electrometer, first the wires in the liquid and then the points in the circuit on either side of the known resistance, was identically the same as that already described as employed in the measurement of the resistance of a film. The required temperature having been reached, the resistance in the box of coils was adjusted to give an electrometer deflection not very different from that of the liquid. The potential difference of the points a, a' was then examined, the battery circuit being broken meanwhile. As in the case of the films this difference was found to be very irregular both in magnitude and direction. As a rule the deflection produced by it did not exceed 2 divisions of the scale, but occasionally it rose to as much as 6 or 7 divisions. The circuit was then completed and after an interval of 3 or 4 minutes, when the rate of weakening of the current (due to polarisation, &c.) appeared to be uniform, the electrometer readings right and left were taken

(1) When the liquid was connected with the electrometer.
(2) ,, resistance ,, ,, ,,
(3) ,, liquid again ,, ,, ,,

The circuit was now broken, and the potential difference of the points a, a' again determined. The temperature at the beginning and end of each set of observations (which occupied about four minutes) was noted. Two sets of observations were always made at each temperature. The mean liquid-deflection derived from (1) and (3) was then divided by the resistance-deflection, after the proper corrections (already described) had been applied to both, and the quotient multiplied by the value of the resistance in the box of coils.

Calculation, based on Joule's law, showed that the rise of temperature of the liquid in the tube, due to the passage of the current, could not in an extreme case, supposing the current to flow continuously for 10 minutes, exceed the one-thirtieth of a degree Centigrade.

A series of experiments was carried out with the view of determining how far the resistance of liquids measured by the electrometer method were in agreement with those obtained by other methods. Since the determinations of the electrical conductivities of liquid, made by Kohlrausch, and by Kohlrausch and Grotrian are more complete, and their results more consistent and trustworthy than those of other observers, it was resolved to test the electrometer method by comparing the resistance of a liquid furnished by it with that of the same liquid given by Kohlrausch. Sulphuric acid was selected for the purpose, and three solutions were prepared, their specific gravities at 18° being 1·1993, 1·4119, and 1·611. A new tube was made, the radius of which was about 0·15 centim. Pieces of the larger tubing were fused to the ends for the insertion of electrodes and thermometers. The distance between the small holes made in it for the reception of thermometers was 12·742 centims., and the constant $\frac{\pi r^2}{l}$ was determined to be 0·00563376.

I. Sulphuric acid, spec. grav. at 18° $=1·1993$, computed from determinations made at 12°·5 and 17°.

Temperature.	Specific resistance ($=\rho$).
Centigrade.	Ohms per cub. centim.
15·38	1·48235
20·61	1·36730
21·46	1·32820
22·78	1·32450
26·26	1·24865

These values were plotted down on a piece of curve paper. From the curve so obtained the specific resistance at 18° was found to be 1·4198.

Kohlrausch (Pogg. Ann., Bd. 159, p. 233) gives for sulphuric acid

$k_{18}10^8=6701$, spec. grav. at 18°=1·1799
$k_{18}10^8=6911$ „ „ =1·2201

where k_{18} is the conductivity at 18° C. referred to that of mercury at 0° as unity.

Assuming that, between the two specific gravities given above, the change in conductivity is proportional to the change in specific gravity, we have

$k_{18}10^8=6802$, for sulphuric acid of spec. grav. 1·1993 at 18°.

hence

$$\rho=\frac{1}{k_{18}}\times\text{specific resistance of mercury}$$
$$=\frac{10^8}{6802}\times\cdot970=1\cdot4260.$$

The value assumed here for the specific resistance of mercury, viz.: ·970, is that given by KOHLRAUSCH himself. MATTHIESSEN gives 0·9619 (EVERETT's 'Physical Units'), but it was thought desirable in transforming KOHLRAUSCH's numbers for liquid conductivities to make use of the specific resistance of mercury which he had himself obtained.

II. SULPHURIC acid, spec. grav. at 18°=1·4119, computed from measurements made at 12°·5 and 16°·5.

The resistance was measured at four temperatures, and from the curve so obtained the specific resistance at 18° was deduced =2·001.

KOHLRAUSCH gives

$k_{18}10^8=5112$, spec. grav. at 18°=1·3946
$k_{18}10^8=3494$ „ „ =1·5014

whence

$k_{18}10^8=4849\cdot9$, for spec. grav. 1·4119

$$\therefore\ \rho=\frac{10^8\times\cdot97}{4849\cdot9}=2\cdot000$$

III. SULPHURIC acid, spec. grav. at 18°=1·611.

The resistance was measured at 17°·56 and at 22°·25. Specific resistance at 18° deduced =4·725.

KOHLRAUSCH gives

$k_{18}10^8=2554$, spec. grav. =1·5707
$k_{18}10^8=1823$ „ =1·6315

whence

$k_{18}10^8=2069\cdot5$ for spec. grav. 1·611.

$$\therefore\ \rho=4\cdot687.$$

The experiments with sulphuric acid were not carried further, as they were intended merely to test the trustworthiness and accuracy of the method. Tabulating the above results, we have—

SULPHURIC Acid.

	Spec. grav at 18°.	Specific resistance at 18°.		Difference.
		By electrometer.	Deduced from KOHLRAUSCH's experiments.	Per cent.
I.	1·1993	1·420	1·426	0·14
II	1·4119	2·001	2·000	0·05
III.	1·6110	4·725	4·687	0·80

The agreement of the resistances measured by the electrometer, with the values obtained for similar solutions by KOHLRAUSCH by a totally different method, is sufficiently close to justify us in placing reliance upon this method for the investigation of other liquids.

X. *Relations between the specific resistance, temperature, and constitution of a liquid.*

A number of experiments were made at different times, by the method just described, on standard and derived solutions of different constitutions. Each liquid was observed at several different temperatures, and the results were plotted down in curves. Between the limits of temperature at which the observations on the films were made (17° C. to 22° C.), these curves were very approximately straight lines, and the equation to any of these might be expressed by the formula

$$\rho=\rho_{20}\{1+a(20-t)\}$$

where ρ_{20} is the specific resistance at 20° C.

The quantity a varied only between the limits 0·027 and 0·032 for standard solutions containing 3, 5, and 7 per cent. of salt respectively, and for derived solutions to which not more than 100 per cent. of water had been added. Addition of glycerine to the standard solution increased the value of a, but, within the range of the change of constitution observed in the films, it did not exceed 0·039. The mean value for all the standard solutions studied, viz.: 0·03, was therefore taken as correct for all others differing but little from them in composition. For the five films mentioned in Section XI., which lost considerable quantities of water, a was taken equal to

$$0·03+0·00013\,\delta$$

where δ is the difference between the specific resistances of the solution and of the standard solution from which it was derived.

In the following table a few out of a large number of examples of the accuracy of this formula as applied to the liquid in mass are given :—

Column I. gives the percentage of salt in the standard solution.

Column II. p or the number of parts by volume of water or glycerine added to $100-p$ parts of standard solution to produce the derived solution.

Column III. gives the value of a obtained from the above rules.

The other columns explain themselves.

TABLE IV.

I. x.	II. p	III a	IV ρ_{20}.	V. $t°$.	VI ρ_t (observed).	VII. ρ_t (calculated).
3	0	0·0300	208·1	16·8	229	228
	..	.	..	19·5	211	211
3	30 H_2O	0·0300	176·0	22·5	160	163
5	0	0·0300	135·0	18·5	142	141
..	0	.		22·3	125	126
5	20 H_2O lost	0·0355	177·4	18·2	188	189
.	.	..	..	21·7	167	167
5	25 H_2O lost	0·0390	203·4	14·9	244	244
..	..	.	..	24·5	172	168

It is unnecessary to quote experiments not made near the temperature limits of the observations on the films. With regard to the value of ρ_{20} it was found that the addition of glycerine increased, of water (within certain limits) diminished, the specific resistance of a standard solution. If, however, the quantity of water added exceeded a certain amount, which varied with the percentage of the solution, further additions increased the specific resistance. As in all the instances of considerable change of constitution to be discussed hereafter, the specific resistance was greater than that of the standard solution, while the hygrometer showed that the air was drier than when the precautions to fill it with aqueous vapour at the tension of that given off by the solution were taken, it follows that, on the assumption also to be discussed hereafter that the specific resistance of the film and the liquid in mass are identical, the only case to be considered is that of a solution which has lost water by evaporation.

A soap film, made from a standard solution, which has lost water by evaporation, may be considered as a mixture of another standard solution, containing a greater percentage of salt and of glycerine.

If x and X be the percentages of the first and second standard solutions respectively, p the number of volumes of water lost in 100 of solution, s and g the number of volumes of standard solution and glycerine in 100 parts of the derived solution, of which the altered film is formed, respectively; then since a standard solution contains 2·2 volumes of glycerine to 3 of water

$$X=x\frac{300}{300-5{\cdot}2p},$$

$$g=\frac{2{\cdot}2}{3}\,\frac{100p}{100-p}, \qquad s=100-g$$

Solutions were made up in accordance with the numbers given by these formulæ to represent films composed originally of a 5 per cent. standard solution, which had lost 15, 20, and 25 parts of water respectively. The standard solution used had a specific resistance of 135 at 20°, somewhat lower than that given by other similar solutions which varied from 140 to 143.

It was found that the specific resistances and parts of water lost were connected by the convenient formula

$$p^2=10(\rho-\rho')=10\,\delta$$

where ρ' is the specific resistance of a standard x per cent. solution.

TABLE V.

	$\rho'=135$.	
ρ.	p.	p (calculated).
135	0	0·0
157	15	14·8
177	20	20·5
203	25	26·0

XI. *Results of the experiments.*

Specific resistance of films.—The observations on the liquid in mass showed, as has been already pointed out, that additions of water diminished, and additions of glycerine increased, the specific resistance. In agreement with this a large number of preliminary experiments proved that the specific resistance of a soap film was higher in dry than in damp air, and that the changes due to variations in the hygroscopic state of the atmosphere were very much larger than any which might be due to the diminishing thickness of the film. As our latest experiments were however in several respects the most full and trustworthy, we confine ourselves in this paper to giving an account of them, premising only as the result of our preliminary work, that any large change in the specific resistance of a film might be safely attributed to the absorption or evaporation of water.

The observations were made between September 13 and October 1, 1880. Several films were rejected either because the insulation was found to be faulty, or because irregular masses of the black or white of the first order, which must have deflected the lines of flow, were collected at the top. Twenty-three films were observed without these defects, and on them our conclusions are based. All were made of the same liquid, which contained 5·05 per cent. of salt, and of which the specific resistance at 20° was 140·5.

The temperatures at which experiments were made varied only between 17°·5 and 22°·5, and, in the first instance, the observations were arranged according to the indications of the hygrometer. Thus, the specific resistances of all the films, during the experiments in which the hygrometer stood between 50 and 51, were calculated for a thickness equal or nearly equal to 9×10^{-5} centims., and the mean taken. All the other films were treated in the same way and the results tabulated. The value 9×10^{-5} centims. was chosen because nearly all the films were observed at that thickness.

TABLE VI.

Hygrometer.	Number of films	Mean value of ρ_{20}
48–49	1	167·9
49–50	5	161·6
50–51	8	156·5
51–52	4	153·7
52–53	..	..
54–55	5	145·7

From this table we learn that on the average the specific resistance of the films was lower and nearer that of the liquid in mass, as the hygrometer stood higher. This result had been anticipated.

The method of moistening the air in the case was open to the objection that the air would probably be a little too dry. Water would evaporate from the blotting paper soaked with the liquid, and the air would therefore be in contact with surfaces moistened with a liquid containing a rather larger percentage of glycerine than the soap solution itself. A small tin tray was made to cover as large a part of the base of the chamber containing the film as was consistent with perfect insulation, and filled with soap solution. A surface of liquid, 52 square centims. in area, was thus exposed, and two films were observed with the air moistened with this alone, and without the blotting paper.

With this arrangement the specific resistances were found to be very high and to rapidly increase, as shown below:—

Thickness.	Specific resistance.	
	Film I.	Film II.
14×10^{-5} centims.	178·8	169·4
4×10^{-5} „	201·6	192·0

These compared so unfavourably with observations made when the blotting paper was used, that it was again employed—in conjunction with the tray—a corner of the paper dipping in the liquid. Occasionally if the paper appeared to be getting dry, it was moistened by liquid introduced by means of a fine pipette through a small hole cut for the purpose in the back of the case.

As in spite of these precautions the air would probably remain a little too dry, there was evidently good reason for excluding from the comparison with the liquid in mass all films for which the reading of the hygrometer was low, and 51° was taken as a provisional limit. Another equally important condition of accuracy was that the hygrometric state of the chamber should be constant, *i.e.*, that the hygrometer and thermometer should be steady during the observations. Changes in the hygrometric state of the air might evidently involve the loss or gain of water by the film. We found that ten films satisfied the conditions that neither hygrometer nor thermometer should vary more than 0°·3 while they were under observation, and that of these four were inadmissible, as the hygrometer stood below 51°. The smallest thickness for which all of the remaining six were examined was 9×10^{-5} centims., and the results at that thickness are given below:—

Date and number of film.	h	t	Observed specific resistance.	ρ_{20}
September 27, III. .	53·1	20·6	141·9	145
,, 28, III. .	53·1	18·9	148·2	143
,, 28, I. .	53·1	18·8	151·7	146
,, . 27, II .	53·9	20·9	142·4	146
,, 30, II. .	51·2	17·9	145·8	137
,, 30, I. .	51·1	17·8	151·6	142
				Mean 143

The mean value of ρ_{20} differs only by 1·8 per cent. from that of the liquid in mass, and the probable error is ± 1 or $\pm 0·71$ per cent.

The conclusion to which this result points, viz.: that the specific resistance of a liquid film is the same as that of a mass of the liquid of the same constitution, is greatly strengthened by another line of argument.

A film, produced in an atmosphere charged with aqueous vapour at a tension differing but slightly from that proper to the liquid from which it was formed, would rapidly adjust itself to that tension. The extremely small quantity of vapour which would have to be absorbed or given off before the condition of equilibrium was attained, and the comparatively large changes of temperature which this absorption or emission would produce in a body of such small mass, would both conduce to this result. It might therefore be that a film the constitution of which differed from that of the liquid sufficiently to produce an appreciable effect upon the specific resistance, might nevertheless have reached a stable state during the first few minutes after its formation and while still too thick to admit of any optical observations being made upon it. If, then, the films, the specific resistances of which agreed most closely with that of the liquid in mass, were also those which obeyed OHM's law with the greatest accuracy, this result would evidently point to the conclusion that the variations in specific resistance were due to changes in constitution and not to the diminishing thickness of the film.

To put this to the test the specific resistances of the films were calculated for apparent thicknesses which were multiples of ten millionths of a centimetre above and for multiples of five millionths of a centimetre below 9×10^{-5} centims. The results, which are too long to be given in full, will be sufficiently indicated by the following table. The films are divided into four classes. Those in Class I. satisfy the three conditions as to constancy of hygrometer, constancy of temperature, and height of hygrometer. Those in Class II. satisfy the first two of these conditions, but not the third. Those in Class III. fail to satisfy the third and one at least of the other conditions. In the case of the films included in Class IV. the arrangements as to moistening the air differed in some respects from those above described (pp. 462, 478).

Column I. gives the date on which the film was observed.

Column II. the number of the film.

Column III. the amount by which the ratio of the specific resistance at the least to that at the greatest observed thickness differs from unity.

In each class μ_1 is the mean of these numbers without reference to sign, μ_2 is the algebraical mean. One film (September 29, V.) is omitted from this list. It was purposely observed under conditions (to be presently described) such as first to increase, and then to diminish the specific resistance, and is therefore useless for the purposes of such a comparison

TABLE VII.

Class I.

I.	II.	III.
September 27 .	II	−0·003
„ 27 .	III.	+0·017
„ 28 .	I.	−0·017
„ 28 .	III.	0·000
„ 30 .	I.	−0·017
„ 30 .	II.	−0·005
		μ_1 = 0·010 μ_2 = −0·005

Class II.

I.	II.	III.
September 18 .	I.	+0·022
„ 20	III.	+0·039
„ 23 .	I.	+0·019
		μ_1 = 0·027 μ_2 = +0·027

Class III.

I.	II.	III.
September 18 .	II	+0·050
„ 20	I	+0·171
„ 20 .	II	+0·084
„ 22 .	I	+0·021
„ 27 .	I.	+0·046
		μ_1 = 0·074 μ_2 = +0·074

Class IV.

I.	II.	III.
September 24 .	I.	+0·136
„ 24	II.	+0·169
„ 25 .	I.	−0·029
„ 29 .	I.	+0·073
„ 29 .	II.	+0·128
October 1 . .	I.	+0·003
„ 1 .	II.	+0·024
„ 1 . .	III	+0·116
		μ_1 = 0·085 μ_2 = +0·078

The conclusion to be drawn from this table is obvious. The divergence from OHM'S law increases rapidly as the conditions become less and less favourable to the preservation of the film in a constant state. The amount of the divergence shown by films in the same class varies considerably; but this is accounted for by the fact that the range of thickness observed, the time during which the film lasted, and the amount of change in the hygrometer and thermometer during that time, are very irregular. There is only one film, however, for which the figure in column III. is less than those which occur in Class I. In the case of this film (October 1, I.) none of the usual precautions were taken, but two dishes of dilute sulphuric acid (specific gravity 1·159) were introduced into the inner chamber. The hygrometer and thermometer read 50° and 17°·5 respectively at the commencement of the observations, and each had risen only 0·1 of a degree at the end. The specific resistance of the film at 20° was 153·4. This number differs less from the specific resistance of the liquid in mass than those often obtained, and, combined with the extreme constancy of the thermometer and hygrometer, proves that a state of things was accidentally produced very similar to and of even greater constancy than that attained by the ordinary methods of moistening. With this single exception, however, all the films included in Class I. obey OHM's law better than any of the others.

In order to test this agreement more fully the following method was adopted. The simplest plan for combining the results of a set of films would be to take the means of the specific resistances at each of the thicknesses at which they were observed. In the case of the films in Class I. the range of comparison possible with this method would have been only from 14×10^{-5} to 9×10^{-5} centims., the greatest and least thicknesses at which all the films were observed.

To get over this difficulty the specific resistance obtained at each thickness for any film was divided by that corresponding to a thickness of 12×10^{-5} centims. for that film. If OHM's law were not obeyed, each of the columns of figures thus obtained ought to give increasing or decreasing values of the ratios as the thickness diminished, and this divergence from the law would have been still more strongly marked in the means of the numbers corresponding to any particular thickness.

The results of this calculation are exhibited in the following table.

Column I. gives the apparent thickness in terms of 10^{-5} centims.

Column II. the colour corresponding to this thickness correct to the nearest tenth of a tint.

Column III. the actual thickness.

Columns IV.–IX. the ratios for the six films.

Column X. the means of these ratios.

The date and number of the film and the readings of the hygrometer and thermometer at the beginning and end of the observations are placed at the head of the column which refers to it.

The thickness 12×10^{-5} centims. was selected as the point of comparison, as at that thickness the specific resistances differed least from their mean value.

TABLE VIII.

I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.	X.
			Sept. 27, II	Sept. 27, III	Sept. 28, I.	Sept. 28, III	Sept 30, I.	Sept. 30, II.	
		Hygrometer	53·0 53·1	53·1 53·2	53·2 53·1	53·1 53·2	51·1 51·2	51·2 51·3	Mean.
		Thermometer	20·9 20·6	20·6 20·4	18·8 18·9	18·9 18·7	17·8 17·9	17·9 18·1	
16·0	R (6, 3)	13·30	.	1·014	..	..	1·013	1·012	1·013
15·0	G (6, 6)	12·47	1·002	1·012	..	1·006	1·008	1·003	1·006
14·0	R (5, 9)	11·64	0·999	1·006	1·005	1·010	1·001	1·004	1·004
13·0	R (5, 3)	10·80	1·001	1·006	1·003	1·016	1·001	1·007	1·006
12·0	G (5, 6)	9·97	1·000	1·000	1·000	1·000	1·000	1·000	1·000
11·0	R (4, 9)	9·14	0·994	0·994	0·997	0·992	1·001	1·000	0·996
10·0	R (4, 1)	8·31	0·990	0·987	0·989	0·986	0·998	1·001	0·992
9·0	G (4, 6)	7·48	0·992	0·976	0·987	0·989	0·996	1·011	0·992
8·5	G (4, 1)	7·06	0·994	0·983	..	0·988	.	1·015	0·995
8·0	BR (3, 2)	6·65	0·994	0·988	..	0·984	..	1·010	0·994
7·5	R (3, 2)	6·23	0·983	0·987	..	0·988	..	1·005	0·991
7·0	Y (3, 3)	5·82	0·992	0·985	.	1·006	.	1·004	0·997
6·5	G (3, 4)	5·40	0·999	0·983	..	..	..	1·011	0·998
6·0	B (3, 4)	4·99	0·999	0·992	..	..	..	1·002	0·998
5·5	P (3, 2)	4·57	.	1·010				1·010	1·010
5·0	O (2, 9)	4·16		1·004	.		.	1·007	1·005
4·5	Y (2, 4)	3·74		1·026				1·007	1·016
								Mean	1·001

The constancy of the numbers in the last column of this table is extremely satisfactory. In no case do they differ from their mean value by more than 1·5 per cent., and this result is certainly as good as could be expected when it is remembered that the thickness is measured by a colour estimation only. It is, however, worth while to investigate the outstanding discrepancies a little more closely. The numbers in column X. are a little less for medium than for large or small thicknesses. The reality of such an alternate decrease and increase in the specific resistance is very improbable, while on the other hand the method of correcting the table of colours by the quotients is likely to produce small periodic errors. Each observation on NEWTON'S rings is, however, independent of the rest, and another table (Table IX.) was therefore prepared as follows:—

Column I. gives in terms of the corrected scale the thicknesses corresponding to the colours measured directly by means of NEWTON'S rings.

Column II. gives numbers proportional to the mean specific resistances of the six films in Class I. at those thicknesses; these are deduced approximately from column X. in the above table.

Column III. gives similar numbers calculated with the thicknesses deduced from NEWTON'S rings instead of those given by the corrected table.

TABLE IX.

I	II	III.
15·48	1·009	1·006
14·10	1·004	·998
12·60	1·003	1·008
11·19	·997	1·006
9·64	·992	·999
8·41	·995	1·015
7·10	·996	·997
5·77	1·004	·997
4·54	1·016	1·007

In the last column of this table there is no regular increase or decrease in the value of the specific resistance and the largest numbers are those which correspond to medium thicknesses. A comparison therefore of the two tables seems to place it beyond question that a soap film thicker than $3·74 \times 10^{-5}$ centims. obeys OHM'S law at all events to within one per cent.

The results of the experiments may be summed up as follows :—

(1.) It is difficult to form a soap film under conditions such as to preclude a slight evaporation or absorption of water. The more nearly such conditions are attained the more closely does the specific resistance of the film agree with that of the liquid in mass. The mean specific resistance of six films observed at a real thickness of $7·48 \times 10^{-5}$ centims. under the most favourable circumstances agreed with that of the liquid to 1·8 per cent.

(2.) The same films obeyed OHM'S law with great accuracy and much better than the others. This was tested between thicknesses of $13·3 \times 10^{-5}$ and $3·74 \times 10^{-5}$ centims. Within these limits the specific resistance never differed from its mean value by more than 1·5 per cent., and showed no tendency to increase or decrease regularly as the thickness diminished.

We may therefore conclude that the specific resistance of a soap film thicker than $3·74 \times 10^{-5}$ is independent of the thickness and is equal to that of the liquid from which it is formed.

The experiments afford no indication of an approach to a thickness equal to the diameter of the sphere of molecular attraction, but if QUINCKE'S result be correct they enable us to determine a superior limit to the difference between the specific resistance of the surface and interior of the liquid respectively.

Let us suppose that the thickness of the film (T) is greater than twice the length of the radius of molecular attraction (ϖ). Let R, ρ, and r be the apparent specific resistance of the film, the specific resistance of the liquid in mass, and the mean specific resistance of the surface layer, the thickness of which is equal to the radius of molecular attraction respectively. Then we have

$$\frac{T}{R}=\frac{T-2\varpi}{\rho}+\frac{2\varpi}{r}$$

Now the experiments certainly prove that there is no regular increase or decrease in the specific resistance of the film between the thicknesses of 13×10^{-5} and $3\cdot7\times10^{-5}$ centims. amounting to as much as 3 per cent. of its value. But if we assign to 2ϖ the value found by QUINCKE, about 1×10^{-5} centims., it may readily be shown from the above formula that, if r differed from ρ by as much as 17 per cent., R would vary 3 per cent. between those thicknesses. As no such variation is observed we must conclude either that the radius of molecular attraction is less than QUINCKE's value or that the mean specific resistance of the surface layer, the thickness of which is equal to the radius of molecular attraction, does not differ by more than 17 per cent. from that of the liquid in mass.

It is obvious that the electrical experiments confirm the accuracy of the revised scale of colours. Had NEWTON's scale been used, the numbers analogous to those given in columns II. and III. of Table IX. would have varied from 0·884 to 1·012.

Change of composition of the films.—The fact that films formed under constant hygrometric conditions obey OHM's law having thus been proved, the second part of the enquiry refers to the change of composition which might under other circumstances be produced.

This is illustrated by experiments on five films.

In the case of some of these the observed specific resistance rose as high as 204, indicating, as that of the liquid in mass was only 140·5, a considerable loss of water.

To determine this the specific resistance was first reduced to 20° C. by the formula

$$\rho=\rho_{20}\{1+(0\cdot03+0\cdot00013\delta)(20-t)\}. \quad \text{(See p. 475.)}$$

The number of parts of water lost out of 100 of the original standard solution was then approximately calculated from the formula

$$p^2=10\delta. \quad \text{(See p. 477.)}$$

The loss of 25 parts of water having been proved to increase the refractive index by 0·018, a corrected value of that quantity was obtained from the formula

$$\mu=1\cdot397+0\cdot00072p,$$

which would be very nearly true up to $p=25$.

This number was then used to correct the thickness of the film, and the new value of ρ thus obtained gave a second approximation to the value of p.

The following table (Table X.) refers to films I. and II., September 24, and films I. and II., September 29. In the case of the first two the air in the case was moistened by the tray of liquid alone without the aid of the blotting paper. When the second pair were observed no precautions were taken to moisten the air.

Column I. gives the date and number of the film.

Column II. the apparent mean thickness in terms of 10^{-5} centims.

Column III. the number of minutes (m) since the formation of the film.

Column IV. p or the number of parts by volume of water lost out of 100 parts of the original standard solution.

Column V. the values of $\frac{\Delta p}{\Delta m}$.

TABLE X.

I.	II.	III. m.	IV. p.	V. $\frac{\Delta p}{\Delta m}$
September 24, I. . .	14·0	10·8	19·0	0·13
	8·0	20·8	20·3	0·17
	4·5	38·3	23·3	
September 24, II. . .	14·0	6·1	16·2	0·16
	8·0	15·6	17·7	0·22
	4·5	33·6	21·7	
September 29, I. . .	14·0	9·9	12·5	0·28
	8·0	18·2	14·8	
September 29, II. . .	14·0	6·4	17·3	0·12
	8·0	13·8	18·2	0·25
	4·5	24·7	20·9	

Bearing in mind that the inner surfaces of these films were in the immediate neighbourhood of the liquid contained in the lower cup to the edge of which the cylinder was attached, we may from this table draw the following conclusions :—

(1.) A soap film, the area of the external surface of which is about 45 square centims., enclosed in a space about 3900 cubic centims. in volume, and the inner surface of which is in contact with air permanently nearly saturated at the tension proper to the solution from which it is formed, may in the course of 40 minutes lose 23·3 out of the 57·7 volumes of water originally contained in every 100 volumes of the solution.

(2.) The values of $\frac{\Delta p}{\Delta m}$ increase as the film gets thinner, but in all cases are much smaller than the value of the same quantity calculated from the moment at which the film was formed to that at which it reached a thickness of 14×10^{-5} centims. Hence, as we should expect, the greater part of the loss, whether measured by the total amount of water evaporated or by the change in the constitution of the film, takes place in the first few minutes. As the film thins, however, the evaporating surface of a given quantity of liquid increases, and thus after a time the rate of change of constitution increases too, no doubt again to diminish when the tension given off by the film is approaching equilibrium with that in the air.

Another film (September 29, V.) was in like manner formed when no precautions had been taken to regulate the amount of moisture in the air. When it was about 11¼ minutes old, a dry sheet of blotting paper hung up inside the inner case was damped by water introduced by a pipette through the hole in the back of the apparatus. The film continued to thin and the electrical resistance to increase, but the specific resistance which had been rising immediately began to fall.

In the following table (Table XI.), column I. gives the number of minutes elapsed since the formation of the film, column II. the apparent thickness of the part between the needles, column III., p, which has the same signification as above.

The table illustrates the rapid changes which a film may undergo. Similar experiments were made with many other films, and always with the same result.

TABLE XI.

I. *m*.	II	III *p*.
5·6	14·0	15·8
6·4	13·0	16·6
7·2	12·0	16·9
8·2	11·0	17·7
9·2	10·0	18·5
10·7	9·0	18·2
11·2	Water introduced.	
11·4	8·5	17·8
12·1	8·0	17·3
12·9	7·5	16·5
13·6	7·0	15·6
14·9	6·5	15·0
16·3	6·0	14·7
18·0	5·5	14·4
20·1	5·0	13·9
23·4	4·5	13·4
28·1	4·0	12·1

All experiments on films which are absorbing or losing water are, of course, subject to some uncertainty as to their temperature. These observations however seem to show that the temperature of the film does not differ much from that of the air. While the film was losing water the temperature would be lower than that of the air, and the error thus introduced would exaggerate the loss, while the heating of the film by the absorption of water would exaggerate the gain. We should therefore expect the change of constitution to appear very rapid about the time when the water was introduced, as at that time the temperature of the film would be changing most rapidly. The low value of p found at 10·7 minutes (see Table XI.) makes it difficult to draw any absolutely certain conclusions. If however we suppose that at the moment the blotting-paper was moistened the value of p was as high as 21, the sudden fall to 17·8 would be explained by a change of temperature not much exceeding 1° C. There is therefore no evidence of any considerable change in temperature when a film suddenly ceases to lose and begins to absorb water.

Turning next to the bearing of these observations on the experiments of MM. PLATEAU and LUDTGE, it is evident that arguments based upon a substance subject to such rapid and considerable changes of composition as are indicated by our observations are open to grave suspicion.

It is indeed true that large changes in the constitution of a soap solution are attended with but trifling alterations in its surface tension. We have for instance proved by experiment that the loss of 25 parts of water alters the tension of a 5 per cent. standard solution by less than 3 per cent. But on the other hand it is evident that, under other circumstances than those detailed above, much larger quantities of water might be gained or lost. In our observations the period of the ultimate slackening of the rate of change of constitution was never reached unless the films were formed in air filled with moisture of the right tension by means of the precautions we have described.

The introduction of a small leaden dish of strong sulphuric acid into the case produced the most violent commotion in a film. Streams of liquid circulated round it with great rapidity, it became spotted with white flecks, and generally broke in less than five minutes. The irregular colouring and the short duration alike prevented accurate measurement in these cases, but it is probable the rate of change was far greater than in those above detailed. The bubble observed by M. PLATEAU lasted for three days, and was placed in a beaker containing caustic potash. The inner surface was, on the other hand, in contact with air which was itself in contact with the water in the manometer. Under these circumstances water may have been either lost or absorbed, and the phenomenon sought for, viz.: a change in surface tension, might have been either caused or masked by the changing constitution of the film.

On the whole, then, we think that the study of the electrical resistance of soap films offers better opportunities of investigating their nature than any other method which has been hitherto proposed. It is far more sensitive to slight changes in constitution, and it provides, as it were, a means of analysing the films, such as measurements on no other of their physical properties afford.

It has been suggested that it would be impossible for a soap film to exist, the thickness of part of which is less than twice the radius of molecular attraction. The surface tension being in this part less than in the remainder, rupture, it is contended, must ensue. This argument might perhaps be conclusive if it were known that the difference of surface tension was considerable. If however it is small while the surface viscosity is high, the force thus set up might for a long time only accelerate the thinning of the film without actually tearing it asunder. In any case while soap films present so many paradoxical phenomena, such as the sudden augmentation of thickness noticed by us in the paper already referred to, it seems unsafe to rest satisfied with conclusions for which direct experimental evidence cannot be adduced.

In conclusion we must express our thanks to Mr. A. HADDON, Demonstrator in the Physical Laboratory of the Royal Naval College, Greenwich, for the ready and able assistance which he has rendered to us during the whole of our investigation.

Our acknowledgments are also due to the late Mr. F. WENTWORTH and to Mr. C. C. STARLING, both of the Yorkshire College, for aid in the tedious calculations required for the revision of the table of colours.

IX. *On the Tidal Friction of a Planet attended by several Satellites, and on the Evolution of the Solar System.*

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[PLATES 61–63.]

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Introduction.

IN previous papers on the subject of tidal friction* I have confined my attention principally to the case of a planet attended by a single satellite. But in order to make the investigation applicable to the history of the earth and moon it was necessary to take notice of the perturbation of the sun. In consequence of the largeness of the sun's mass it was not there requisite to make a complete investigation of the theory of a planet attended by a pair of satellites.

In the first part of this paper the theory of the tidal friction of a central body attended by any number of satellites is considered.

In the second part I discuss the degree of importance to be attached to tidal friction as an element in the evolution of the solar system and of the several planetary subsystems.

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* Phil. Trans, Parts I. and II., 1879, and Part II. 1880; Proc. Roy. Soc., No. 197, 1879, and No. 202, 1880.

The last paragraph contains a discussion of the evidence adduced in this part of the paper, and a short recapitulation of the observed facts in the solar system which bear on the subject. This is probably the only portion which will have any interest for others than mathematicians.

I.

THE THEORY OF THE TIDAL FRICTION OF A PLANET ATTENDED BY ANY NUMBER OF SATELLITES.

§ 1. *Statement and limitation of the problem.*

Suppose there be a planet attended by any number of satellites, all moving in circular orbits, the planes of which coincide with the equator of the planet; and suppose that the satellites all raise tides in the planet. Then the problem proposed for solution is to investigate the gradual changes in the configuration of the system under the influence of tidal friction.

This problem is only here treated under certain restrictions as to the nature of the tidal friction and in other respects. These limitations however will afford sufficient insight into the more general problem. The planet is supposed to be a homogeneous spheroid formed of viscous fluid, and the only case considered in detail is that where the viscosity is small; moreover, in the tidal theory adopted the effects of inertia are neglected. I have however shown elsewhere that this neglect is not such as to materially vitiate the theory.* The satellites are treated as attractive particles which have the power of attracting and being attracted by the planet, but have no influence upon one another. A consequence of this is that each satellite only raises a single tide in the planet, and that it is not necessary to take into consideration the actual distribution of the satellites at any instant of time. We are thus only concerned in determining the changes in the distances of the satellites and in the rotation of the planet.

If the mutual perturbation of the satellites were taken into account the problem would become one of the extremest complication. We should have all the difficulties of the planetary theory in determining the various inequalities, and, besides this, it would be then necessary to investigate an indefinitely long series of tidal disturbances induced by these inequalities of motion, and afterwards to find the secular disturbances due to the friction of these tides.

It is however tolerably certain that in general these inequality-tides will exercise a very small influence compared with that of the primary tide. Supposing a relationship between the mean motions of two, three, or more satellites, like that which holds good in the Jovian system, to exist at any epoch, then it is not credible but that such relationship should be broken down in time by tidal friction. General considerations would lead one to believe that the first effect of tidal friction would be to set up amongst the satellites in question an oscillation of mean motions about the average values which satisfy the supposed definite relationship; afterwards this oscillation would go on increasing indefinitely until a critical state was reached in which the average mean motions would break loose from the relationship, and the oscillation would subsequently die away. It seems probable therefore that in the history of such a system there would be a series of periods during which the mutual perturbations of the satellites would exercise a considerable but temporary effect, but that on the whole the system would change nearly as though the satellites exercised no mutually perturbing power.

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* "Problems Connected with the Tides of a Viscous Spheroid—No. III.," Phil. Trans., Part. II., 1879.

There is however one case in which mutual perturbation would probably exercise a lasting effect on the system. Suppose that in the course of the changes two satellites came to have nearly the same mean distance, then these two bodies might either come ultimately into collision or might coalesce so as to form a double system like that of the earth and moon, which revolve round the sun in the same period. In this paper I do not make any attempt to trace such a case, and it is supposed that any satellite may pass freely through a configuration in which its distance is equal to that of any other satellite.

§ 2. *Formation and transformation of the differential equations.*

In this paper I shall have occasion to make frequent use of the idea of moment of momentum. This phrase is so cumbrous that I shall abridge it and speak generally of angular momentum, and in particular of rotational momentum and orbital momentum when meaning moment of momentum of a planet's rotation and moment of momentum of the orbital motion of a satellite. I shall also refer to the principle of conservation of moment of momentum as that of conservation of momentum.

The notation here adopted is almost identical with that of previous papers on the case of the single satellite and planet; it is as follows:—

For the planet let:

M=mass; a=mean radius; g=mean pure gravity; w=mass per unit volume; υ=viscosity; $\mathfrak{g}=\frac{2}{5}g/a$; n=angular velocity of rotation; C=moment of inertia about the axis of rotation, and therefore, neglecting the ellipticity of figure, equal to $\frac{2}{5}Ma^2$.

For any particular one of the system of satellites, let:

m=mass; c=distance from planet's centre; Ω=orbital angular velocity.

Also μ being the attraction between unit masses at unit distance, let $\tau=\frac{3}{2}\mu m/c^3$; and let $\nu=M/m$.

These same symbols will be used with suffixes 1, 2, 3, &c., when it is desired to refer to the 1st, 2nd, 3rd, &c., satellite, but when (as will be usually the case) it is desired simply to refer to any satellite, no suffixes will be used.

Where it is necessary to express a summation of similar terms, each corresponding to one satellite, the symbol Σ will be used; *e.g.*, $\Sigma\kappa c^{\frac{1}{2}}$ will mean $\kappa_1 c_1^{\frac{1}{2}}+\kappa_2 c_2^{\frac{1}{2}}+$ &c.

Now consider the single satellite m, c, Ω, &c.

If this satellite alone were to raise a tide in the planet, the planet would be distorted into an ellipsoid with three unequal axes, and in consequence of the postulated internal friction, the major axis of the equatorial section of the planet would be directed to a point somewhat in advance of the satellite in its orbit.

Let f be the angle made by this major axis with the satellite's radius vector; f is then a symbol subject to suffixes 1, 2, 3, &c., because it will be different for each satellite of the system.

Then it is proved in (22) of my paper on the "Precession of a Viscous Spheroid,"* that the tidal frictional couple due to this satellite's attraction is $C\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}$.

Now it appears from Sec. 14 of the same paper that the tidal reaction, which affects the motion of each satellite, is independent of the tides raised by all the other satellites.

Hence the principle of conservation of momentum enables us to state, that the rate of increase of the orbital momentum of any satellite is equal to the rate of the loss of rotational momentum of the planet which is caused by that satellite alone. The rate of loss of this latter momentum is of course equal to the above tidal frictional couple.

When the planet is reduced to rest the orbital momentum of the satellite in the circular orbit is $\Omega c^2Mm/(M+m)$. Hence the equation of tidal reaction, which gives the rate of change in the satellite's distance, is

$$\frac{d}{dt}\left[\frac{Mm}{M+m}\Omega c^2\right]=C\frac{1}{2}\frac{\tau^2}{\mathfrak{g}}\sin 4\mathrm{f}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

A similar equation will hold true for each satellite of the system.

This equation will now be transformed.

By KEPLER's law $\Omega^2c^3=\mu(M+m)$ and therefore

$$\frac{Mm}{M+m}\Omega c^2=\mu^{\frac{1}{2}}\frac{Mm}{(M+m)^{\frac{1}{2}}}c^{\frac{1}{2}}$$

By the theory of the tides of a viscous spheroid (Phil. Trans., Part I., 1879, p. 13)

$$\tan 2\mathrm{f}=\frac{2(n-\Omega)}{2\mathfrak{p}},\text{ where } 2\mathfrak{p}=\frac{2gaw}{19\upsilon}$$

Hence $$\sin 4\mathrm{f}=\frac{2(n-\Omega)/\mathfrak{p}}{1+(n-\Omega)^2/\mathfrak{p}^2},\text{ also } \tau^2=\left(\tfrac{3}{2}\right)^2\frac{\mu^2m^2}{c^6}$$

Hence (1) becomes

$$\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}\frac{dc^{\frac{1}{2}}}{dt}=\left(\tfrac{3}{2}\right)^2\frac{C}{\mathfrak{g}}\frac{(\mu m)^2}{c^6}\frac{(n-\Omega)/\mathfrak{p}}{1+(n-\Omega)^2/\mathfrak{p}^2}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Now let Ch be the angular momentum of the whole system, namely that due to the planet's rotation and to the orbital motion of all the satellites. And let CE be the whole energy, both kinetic and potential, of the system. Then h is the angular velocity with which the planet would have to rotate in order that the rotational momentum might be equal to that of the whole system; and E is twice the square of the angular velocity with which the planet would have to rotate in order that the kinetic energy of planetary rotation might be equal to the whole energy of the system. By the principle of conservation of momentum h is constant, and since the system is non-conservative of energy E is variable, and must diminish with the time.

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* Phil. Trans., Part II., 1879, p. 459.

The kinetic energy of the orbital motion of the satellite m is $\frac{1}{2}\mu Mm/c$, and the potential energy of position of the planet and satellite is $-\mu Mm/c$; the kinetic energy of the planet's rotation is $\frac{1}{2}Cn^2$. Thus we have,

$$Ch=Cn+\Sigma\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}c^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$2CE=Cn^2-\Sigma\frac{\mu Mm}{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the equations (3) and (4) we may regard C as a constant, provided we neglect the change of ellipticity of the planet's figure as its rotation slackens.

Let the symbol δ indicate partial differentiation; then from (3) and (4)

$$\frac{\delta n}{\delta(c^{\frac{1}{2}})}=-\frac{1}{C}\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}$$

$$-\frac{\delta E}{\delta(c^{\frac{1}{2}})}=\frac{1}{C}\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}n-\frac{1}{C}\frac{\mu Mm}{c^{\frac{3}{2}}}$$

But
$$\frac{1}{C}\frac{\mu Mm}{c^{\frac{3}{2}}}=\frac{1}{C}\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}\Omega$$

and therefore
$$-\frac{C(M+m)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}Mm}\frac{\delta E}{\delta(c^{\frac{1}{2}})}=n-\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From equations (2) and (5) we may express the rate of increase of the square root of any satellite's distance in terms of the energy of the whole system, in the general case where the planet has any degree of viscosity. A good many transformations, analogous to those below, may be made in this general case, but as I shall only examine in detail the special case in which the viscosity is small, it will be convenient to make the transition thereto at once.

When the viscosity is small, $\mathfrak{p}$, which varies inversely as the viscosity, is large. Then, unless $n-\Omega$ be very large, $(n-\Omega)/\mathfrak{p}$ is small compared with unity. Thus in (2) we may neglect $(n-\Omega)^2/\mathfrak{p}^2$ in the denominator compared with unity.

Substituting from (5) in (2), and making this approximation, we have

$$\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}\frac{dc^{\frac{1}{2}}}{dt}=-\left(\tfrac{3}{2}\right)^2\frac{C}{\mathfrak{g}}\frac{(m\mu)^2}{c^6}\frac{C(M+m)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}Mm\mathfrak{p}}\frac{\delta E}{\delta(c^{\frac{1}{2}})} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now let
$$\xi=\left(\frac{M}{M+m}\right)^{\frac{1}{2}}\left(\frac{c}{\alpha}\right)^{\frac{7}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where α is any constant length, which it may be convenient to take either as equal to the mean radius of the planet, or as the distance of some one of the satellites at some fixed epoch. ξ is different for each satellite and is subject to the suffixes 1, 2, 3, &c.

The equation (6) may be written

$$\left(\frac{M}{M+m}\right)^{\frac{1}{2}}7c^3\frac{dc^{\frac{1}{2}}}{dt}=-\left(\tfrac{3}{2}\right)^2\times 49\times\frac{\mu C^2}{M\mathfrak{g}\mathfrak{p}}\times\left(\frac{M+m}{M}\right)^{\frac{1}{2}}\frac{1}{7c^3}\,\frac{\delta E}{\delta(c^{\frac{1}{2}})}$$

Now let A
$$A=\left(\tfrac{3}{2}\right)^2\times 49\times\frac{\mu C^2}{M\alpha^7\mathfrak{g}\mathfrak{p}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

And we have
$$\frac{d\xi}{dt}=-A\frac{\delta E}{\delta\xi} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

[In order to calculate A it may be convenient to develop its expression further.

$$M\mathfrak{g}\alpha^7=\tfrac{2}{5}\frac{\mu M}{a^3}\cdot M\alpha^7=\tfrac{5}{2}\mu C^2\left(\frac{\alpha}{a}\right)^7\text{, so that }\frac{\mu C^2}{M\mathfrak{g}\alpha^7}=\tfrac{2}{5}\left(\frac{a}{\alpha}\right)^7$$

and
$$A=\left(\tfrac{3}{2}\right)^2\left(\tfrac{2}{5}\right)49\frac{(a/\alpha)^7}{\mathfrak{p}}\text{, where }\mathfrak{p}=\frac{ga w}{19v} \quad . \quad . \quad . \quad . \quad (10)$$

Since $\mathfrak{p}$ is an angular velocity A is a period of time, and A is the same for all the satellites.]

In (9) ξ is the variable, but it will be convenient to introduce an auxiliary variable x, such that

or
$$\left.\begin{aligned}x^7&=\xi\\ x&=\left(\frac{M}{M+m}\right)^{\frac{1}{14}}\left(\frac{c}{\alpha}\right)^{\frac{1}{2}}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Then
$$\frac{\mu^{\frac{1}{2}}Mm}{(M+m)^{\frac{1}{2}}}c^{\frac{1}{2}}=\frac{\mu^{\frac{1}{2}}M^{\frac{13}{14}}m\alpha^{\frac{1}{2}}}{(M+m)^{\frac{3}{7}}}x$$

Let
$$\kappa=\frac{\mu^{\frac{1}{2}}M^{\frac{13}{14}}m\alpha^{\frac{1}{2}}}{C(M+m)^{\frac{3}{7}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

κ is different for each satellite and is subject to suffixes 1, 2, 3, &c.

Thus (3) may be written
$$h=n+\Sigma\kappa x \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Again
$$\frac{\mu Mm}{c}=\frac{\mu M^{\frac{8}{7}}m}{(M+m)^{\frac{1}{7}}\alpha}\,\frac{1}{x^2}$$

Let
$$\lambda=\frac{\mu M^{\frac{8}{7}}m}{C(M+m)^{\frac{1}{7}}\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

λ is different for each satellite and is subject to suffixes 1, 2, 3, &c.

On comparing (12) and (14) we see that

$$\frac{\lambda}{\kappa}=\Omega x^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

This is of course merely a form of writing the equation

$$\mu(M+m)=\Omega^2c^3$$

Then (4) may be written

$$2E=n^2-\Sigma\frac{\lambda}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

[In order to compute κ and λ we may pursue two different methods.

First, suppose $\alpha=a$, the planet's mean radius.

Then
$$\frac{m\alpha^{\frac{1}{2}}}{C}=\frac{5}{2\nu a^{\frac{3}{2}}};\ M^{\frac{7}{14}}\mu^{\frac{1}{2}}=(ga^2)^{\frac{1}{2}};\ \frac{M^{\frac{6}{14}}}{(M+m)^{\frac{3}{7}}}=\left(\frac{\nu}{1+\nu}\right)^{\frac{3}{7}}$$

$\kappa=\frac{5}{2}[\nu^4(1+\nu)^3]^{-\frac{1}{7}}\left(\frac{g}{a}\right)^{\frac{1}{2}}$, of same dimensions as an angular velocity.

$\lambda=\frac{5}{2}[\nu^6(1+\nu)]^{-\frac{1}{7}}\left(\frac{g}{a}\right)$, of same dimensions as the square of an angular velocity.

If ν be large compared with unity, as is generally the case, the expressions become

$$\kappa=\frac{5m}{2M}\sqrt{\frac{g}{a}}, \qquad \lambda=\frac{5m}{2M}\left(\frac{g}{a}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

Secondly, suppose M large compared with all the m's, and suppose for example that the solar system as a whole is the subject of investigation. Then take α as the earth's present radius vector, and ω as its present mean motion, and

$$\kappa=\frac{m}{C}\sqrt{\mu M\alpha}, \quad \text{and } \lambda=\frac{m}{C}\frac{\mu Mm}{a}$$

$$\text{or} \quad \kappa=m\left(\frac{\omega\alpha^2}{C}\right), \qquad \lambda=m\left(\frac{\omega^2\alpha^2}{C}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

C is here the sun's moment of inertia.]

Then collecting results from (9), (13), (16), the equations which determine the changes in the system are

$$\left.\begin{array}{l} \dfrac{d\xi}{dt}=-A\dfrac{\delta E}{\delta\xi} \\ \text{and a similar equation for each satellite} \\ n=h-\Sigma\kappa x \\ 2E=n^2-\Sigma\dfrac{\lambda}{x^2} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

where $x^7=\xi$; A is a certain time to be computed as above shown in (10); κ an angular velocity to be computed as above shown in (17) and (18); and λ the square of an angular velocity to be computed as above in (17) and (18).

Also
$$\xi=\left(\frac{M}{M+m}\right)^{\frac{7}{2}}\left(\frac{c}{a}\right)^{\frac{7}{2}}=\left(\frac{\nu}{1+\nu}\right)^{\frac{7}{2}}\left(\frac{c}{a}\right)^{\frac{7}{2}}$$

If ν be large compared with unity, ξ is very approximately proportional to the seventh power of the square root of the satellite's distance.

The solution of this system of simultaneous differential equations would give each of the ξ's in terms of the time; afterwards we might obtain n and E in terms of the time from the last two of (19).

These differential equations possess a remarkable analogy with those which represent HAMILTON'S principle of varying action (THOMSON and TAIT'S 'Nat. Phil.,' 1879, § 330 (14)).

The rate of loss of energy of the system may be put into a very simple form. This function has been called by Lord RAYLEIGH ('Theory of Sound,' vol. i., § 81) the Dissipation Function,* and the name is useful, because this function plays an important part in non-conservative systems.

In the present problem the Dissipation Function or Dissipativity is $-C\frac{dE}{dt}$.

Now
$$\frac{dE}{dt}=\Sigma\frac{\delta E}{\delta\xi}\frac{d\xi}{dt}$$

From (19) the dissipativity is therefore either

$$CA\Sigma\left(\frac{\delta E}{\delta\xi}\right)^2 \text{ or } \frac{C}{A}\Sigma\left(\frac{d\xi}{dt}\right)^2$$

This quantity is of course essentially positive.

It is easy to show that $\frac{\delta E}{\delta\xi}=-\frac{\kappa}{7x^6}(n-\Omega)$

Then on substituting for the various symbols in the expression for the dissipativity their values in terms of the original notation, we have

$$-\frac{dE}{dt}=\Sigma\frac{\tau^2}{\mathfrak{gp}}(n-\Omega)^2$$

Or if N be the tidal frictional couple corresponding to the satellite m,

$$-C\frac{dE}{dt}=\Sigma N(n-\Omega)$$

This last result would be equally true whatever were the viscosity of the planetary spheroid.

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* Sir W. THOMSON prefers to modify the name by calling it Dissipativity.

The dissipativity, converted into heat by JOULE's equivalent, expresses the amount of heat generated per unit time within the planetary spheroid. This result has been already obtained in a different manner for the case of a single satellite in a previous paper ("Problems, &c.," Phil. Trans., Part II., 1879, p. 557).

§ 3. *Sketch of method for solution of the equations by series.*

It does not seem easy to obtain a rigorous analytical solution of the system (19) of differential equations. I have however solved the equations by series, so as to obtain analytical expressions for the ξ's, as far as the fourth power of the time. This solution is not well adapted for the purposes of the present paper, because the series are not rapidly convergent, and therefore cannot express those large changes in the configuration of the system which it is the object of the present paper to trace.

As no subsequent use is made of this solution, and as the analysis is rather long, I will only sketch the method pursued.

If $\frac{1}{49}A$ be taken as the unit of time $\frac{dE}{dt}=-\frac{1}{49}\Sigma\left(\frac{\delta E}{\delta\xi}\right)^2$

Differentiating again and again with regard to the time, and making continued use of this equation, we find d^2E/dt^2, d^3E/dt^3, &c., in terms of $\delta E/\delta\xi$.

It is then necessary to develop these expressions by performing the differentiations with regard to ξ.

An abridged notation was used in which $\left[\frac{\mathrm{a},\ \mathrm{b}}{\mathrm{p}}\right]^{\mathrm{k}}$ represented $\left(\frac{\mathrm{a}\lambda x^{-3}-\mathrm{b}n\kappa}{x^{\mathrm{p}}}\right)^{\mathrm{k}}$ or $\left[\frac{\kappa(\mathrm{a}\Omega-\mathrm{b}n)}{x^{\mathrm{p}}}\right]^{\mathrm{k}}$. With this notation the whole operation may be shown to depend on the performance of $\delta/\delta\xi$ on expressions of the form

$$\Sigma\gamma\left[\frac{\mathrm{a}_1,\ \mathrm{b}_1}{\mathrm{p}_1}\right]^{\mathrm{k}_1}\left[\frac{\mathrm{a}_2,\ \mathrm{b}_2}{\mathrm{p}_2}\right]^{\mathrm{k}_2}\cdots\left[\frac{\mathrm{a}_r,\ \mathrm{b}_r}{\mathrm{p}_r}\right]^{\mathrm{k}_r}$$

where γ is independent of ξ, but may be a function of the mass of each satellite.

Having evaluated the successive differentials of E we have

$$E=E_0+t\left(\frac{dE}{dt}\right)_0+\frac{t^2}{1.2}\left(\frac{d^2E}{dt^2}\right)_0+\frac{t^3}{1.2.3}\left(\frac{d^3E}{dt^3}\right)_0+\text{ \&c.}$$

Where the suffix 0 indicates that the value, corresponding to $t=0$, is to be taken.

It is also necessary to evaluate the successive differentials of $\delta E/\delta\xi$ with regard to the time, and then we have

$$\xi=\xi_0-t\left(\frac{\delta E}{\delta\xi}\right)_0-\frac{t^2}{1.2}\left(\frac{\delta}{\delta\xi}\frac{dE}{dt}\right)_0-\frac{t^3}{1.2.3}\left(\frac{\delta}{\delta\xi}\frac{d^2E}{dt^2}\right)_0-\text{ \&c.}$$

The coefficient of t^4 was found to be very long even with the abridged notation, and involved squares and products of Σ's.

§ 4. *Graphical solution in the case when there are not more than three satellites.*

Although a general analytical solution does not seem attainable, yet the equations have a geometrical or quasi-geometrical meaning, which makes a complete graphical solution possible, at least in the case where there are not more than three satellites.

To explain this I take the case of two satellites only, and to keep the geometrical method in view I change the notation, and write z for E, x for ξ_1, and y for ξ_2, also I write Ω_x for Ω_1, and Ω_y for Ω_2. The unit of time is chosen so that $A=1$.

Then the equations (19) become

$$\frac{dx}{dt}=-\frac{\delta z}{\delta x},\quad \frac{dy}{dt}=-\frac{\delta z}{\delta y} \quad \dots\dots\dots\dots \quad (20)$$

and

$$2z=(h-\kappa_1 x^{\frac{1}{2}}-\kappa_2 y^{\frac{1}{2}})^2-\frac{\lambda_1}{x^{\frac{1}{2}}}-\frac{\lambda_2}{y^{\frac{1}{2}}} \quad \dots\dots\dots\dots \quad (21)$$

Now suppose a surface constructed to illustrate (21), x, y, z being the coordinates of any point on it. Let the axes of x and y be drawn horizontally, and that of z vertically upwards. The z ordinate of course gives the energy of the system corresponding to any values of x and y which are consistent with the given angular momentum h:

We have for the dissipativity of the system

$$-\frac{dz}{dt}=\left(\frac{\delta z}{\delta x}\right)^2+\left(\frac{\delta z}{\delta y}\right)^2$$

Whence

$$\frac{dx}{dz}=\frac{\frac{\delta z}{\delta x}}{\left(\frac{\delta z}{\delta x}\right)^2+\left(\frac{\delta z}{\delta y}\right)^2},\quad \frac{dy}{dz}=\frac{\frac{\delta z}{\delta y}}{\left(\frac{\delta z}{\delta x}\right)^2+\left(\frac{\delta z}{\delta y}\right)^2} \quad \dots\dots\dots \quad (22)$$

Let $(X-x)/\lambda=(Y-y)/\mu=Z-z$ be the equations to a straight line through a point x, y, z on the surface. Then if this line lies in the tangent plane at that point

$$\lambda\frac{\delta z}{\delta x}+\mu\frac{\delta z}{\delta y}-1=0.$$

The inclination of this line to the axis of z will be a maximum or minimum when $\lambda^2+\mu^2$ is a maximum or minimum. In other words if this straight line is a tangent line to the steepest path through x, y, z on the surface, $\lambda^2+\mu^2$ must be a maximum or minimum.

Hence for this condition to be fulfilled we must have

$$\lambda\delta\lambda+\mu\delta\mu=0$$

$$\frac{\delta z}{\delta x}\delta\lambda+\frac{\delta z}{\delta y}\delta\mu=0$$

And therefore $\lambda/\frac{\delta z}{\delta x}=\mu/\frac{\delta z}{\delta y}$, and these are both equal to $1\Big/\left\{\left(\frac{\delta z}{\delta x}\right)^2+\left(\frac{\delta z}{\delta y}\right)^2\right\}$

Therefore the equation to the tangent to the steepest path is

$$\frac{X-x}{\partial z/\partial x}=\frac{Y-y}{\partial z/\partial y}=\frac{Z-z}{(\partial z/\partial x)^2+(\partial z/\partial y)^2}\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\qquad (23)$$

If this steepest path on the energy surface is the path actually pursued by the point which represents the configuration of the system, equation (23) must be satisfied by

$$X=x+\frac{dx}{dz}\delta z,\quad Y=y+\frac{dy}{dz}\delta z,\quad Z=z+\delta z$$

And therefore we must have

$$\frac{dx}{dz}=\frac{\frac{\partial z}{\partial x}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2},\quad \frac{dy}{dz}=\frac{\frac{\partial z}{\partial y}}{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

But these are the values already found in (22) for dx/dz and dy/dz.

Therefore we conclude that the representative point always slides down a steepest path on the energy surface. Hence it only remains to draw the surface, and to mark out the lines of steepest slope in order to obtain a complete graphical solution of the problem. Since the lines of greatest slope cut the contours at right angles, if we project the contours orthogonally on to the plane of $x\,y$, and draw the system of orthogonal trajectories of the contours, we obtain a solution in two dimensions. This solution will be exhibited below, but for the present I pass on to more general considerations.

A precisely similar argument might be applied to the case where there are any number of satellites, only as space has but three dimensions, a geometrical solution is not possible. If there be r satellites, then the problem to be solved may be stated in geometrical language thus :—

It is required to find the path which is inclined at the least angle to the axis of E on the locus

$$2E=(h-\Sigma\kappa\xi^{\frac{1}{2}})^2-\Sigma\frac{\lambda}{\xi^{\frac{1}{2}}}$$

This locus is described in space of $r+1$ dimensions. One axis is that of E, and the remaining r axes are the axes of the r different ξ's. The solution may be depressed so as to merely require space of r dimensions, for we may, in space of r dimensions, construct the orthogonal trajectories of the contour loci found by attributing various values to E.

Thus we might actually solve geometrically the case of three satellites. The energy locus here involves space of four dimensions, but the contour loci are a family of surfaces in three dimensions. If such a system of surfaces were actually constructed, it would be possible to pass through them a number of wires or threads which should be a good approximation to the orthogonal trajectories. The trouble of execution would however be hardly repaid by the results, because most of the interesting general conclusions may be drawn from the case of two satellites, where we have only to deal with curves.

If the case of a single satellite be considered, we see that the energy locus is a curve, and the transit along the steepest path degenerates merely into travelling down hill. Now as the slopes of the energy curve are not altered in direction, but merely in steepness, by taking the abscissas of points on the curve as any power of ξ, the solution may still be obtained if we take x (or $\xi^{\frac{1}{4}}$) as the abscissa instead of ξ. This reduces the solution to exactly that which was given in a previous paper, where the graphical method was applied to the case of a single satellite.*

§ 5. *The graphical method in the case of two satellites.*

I now return to the special case in which there are only two satellites. The equation to the surface of energy is given in (21). The maxima and minima values of z (if any) are given by equating $\partial z/\partial x$ and $\partial z/\partial y$ to zero. This gives

$$\left.\begin{aligned} h-\kappa_1 x^{\frac{1}{4}}-\kappa_2 y^{\frac{1}{4}} &= \frac{\lambda_1}{\kappa_1}\frac{1}{x^{\frac{3}{4}}} \\ h-\kappa_1 x^{\frac{1}{4}}-\kappa_2 y^{\frac{1}{4}} &= \frac{\lambda_2}{\kappa_2}\frac{1}{y^{\frac{3}{4}}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

By (15) and (19) we see that these equations may be written

$$\left.\begin{aligned} n &= \Omega_x \\ n &= \Omega_y \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

They also lead to the equations

$$\left.\begin{aligned} (x^{\frac{1}{4}})^4-\frac{h-\kappa_2 y^{\frac{1}{4}}}{\kappa_1}(x^{\frac{1}{4}})^3+\frac{\lambda_1}{\kappa_1{}^2} &= 0 \\ (y^{\frac{1}{4}})^4-\frac{h-\kappa_1 x^{\frac{1}{4}}}{\kappa_2}(y^{\frac{1}{4}})^3+\frac{\lambda_2}{\kappa_2{}^2} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Now an equation of the form $Y^4-\alpha Y^3+\beta=0$ may be written $(Y\beta^{-\frac{1}{4}})^4-\alpha\beta^{-\frac{1}{4}}(Y\beta^{-\frac{1}{4}})^3+1=0$. And I have proved in a previous paper† that an equation $x^4-hx^3+1=0$ has two real roots, if h be greater than $4/3^{\frac{3}{4}}$, but has no real roots if h be less than $4/3^{\frac{3}{4}}$. Hence it follows that this equation in Y has two real roots, if α be greater than $4\beta^{\frac{1}{4}}/3^{\frac{3}{4}}$, but no real roots if it be less.

Then if we consider the two equations (26) as biquadratics for $x^{\frac{1}{4}}$ and $y^{\frac{1}{4}}$ respectively, we see that the first has, or has not, a pair of real roots, according as

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* Proc. Roy. Soc., No. 197, 1879.

† Ibid., No. 202, 1880, p. 260–263.

$$h-\kappa_2 y^{\frac{1}{2}} \text{ is greater or less than } \left(\frac{4}{3^{\frac{3}{4}}}\right) \lambda_1^{\frac{3}{4}}\kappa_1^{\frac{1}{4}},$$

and the second has, or has not, a pair of real roots, according as

$$h-\kappa_1 x^{\frac{1}{2}} \text{ is greater or less than } \left(\frac{4}{3^{\frac{3}{4}}}\right) \lambda_2^{\frac{3}{4}}\kappa_2^{\frac{1}{4}}.$$

Now if we substitute for the λ's and κ's their values, we find that

$$\lambda_1^{\frac{3}{4}}\kappa_1^{\frac{1}{4}}=\frac{\mu^{\frac{1}{2}}(Mm_1)^{\frac{3}{4}}}{C^{\frac{3}{4}}(M+m_1)^{\frac{1}{4}}}$$

$$\lambda_2^{\frac{3}{4}}\kappa_2^{\frac{1}{4}}=\text{the same with } m_2 \text{ in place of } m_1.$$

Now let γ_1 and γ_2 be two lengths determined by the equations

$$\frac{Mm_1}{M+m_1}\gamma_1^2=\frac{Mm_2}{M+m_2}\gamma_2^2=C.$$

Or in words—let γ_1 be such a distance that the moment of inertia of the planet (concentrated at its centre) and the first satellite about their common centre of inertia may be equal to the planet's moment of inertia about its axis of rotation; and let γ_2 be a similar distance involving the second satellite instead of the first.

And let ω_1, ω_2 be two angular velocities determined by the equations

$$\omega_1^2\gamma_1^3=\mu(M+m_1), \qquad \omega_2^2\gamma_2^3=\mu(M+m_2).$$

Or in words—let ω_1 be the angular velocity of the first satellite when revolving in a circular orbit at distance γ_1, and ω_2 a similar angular velocity for the second satellite when revolving at distance γ_2.

Now $$\left[\frac{\mu Mm_1}{C}\right]^{\frac{1}{2}}=\omega_1\gamma_1^{\frac{1}{2}} \text{ and } \left[\frac{Mm_1}{C(M+m_1)}\right]^{\frac{1}{4}}=\frac{1}{C}\frac{Mm_1}{M+m_1}\gamma_1^{\frac{3}{2}}.$$

So that $$\lambda_1^{\frac{3}{4}}\kappa_1^{\frac{1}{4}}=\frac{1}{C}\frac{Mm_1}{M+m_1}\omega_1\gamma_1^2,$$

and similarly $\lambda_2^{\frac{3}{4}}\kappa_2^{\frac{1}{4}}$=the same with the suffix 2 in place of 1. Hence the first of the two equations (26) has, or has not, a pair of real roots, according as

$$C(h-\kappa_2 y^{\frac{1}{2}}) \text{ is greater or less than } \frac{4}{3^{\frac{3}{4}}}\frac{Mm_1}{M+m_1}\omega_1\gamma_1^2,$$

and the second has, or has not, a pair of real roots, according as

$$C(h-\kappa_1 x^{\frac{1}{2}}) \text{ is greater or less than } \frac{4}{3^{\frac{3}{4}}}\frac{Mm_2}{M+m_2}\omega_2\gamma_2^2.$$

It is obvious that $Mm_1\omega_1\gamma_1^2/(M+m_1)$ is the orbital momentum of the first satellite when revolving at distance γ_1, and similarly $Mm_2\omega_2\gamma_2^2/(M+m_2)$ is the orbital momentum of the second satellite when revolving at distance γ_2.

If the second or y-satellite be larger than the first or x-satellite the latter of these momenta is larger than the first.

Now Ch is the whole angular momentum of the system, and in order that there may be maxima and minima determined by the equations $\delta z/\delta x=0$, $\delta z/\delta y=0$, the equations (26) must have real roots. Then on putting y equal to zero in the first of the above conditions, and x equal to zero in the second we get the following results:—

First, there are no maxima and minima points for sections of the energy surface either parallel to x or y, if the whole momentum of the system be less than $4/3^{\frac{3}{4}}$ times the orbital momentum of the smaller or x-satellite when moving at distance γ_1.

Second, there are maxima and minima points for sections parallel to x, but not for sections parallel to y, if the whole momentum be greater than $4/3^{\frac{3}{4}}$ times the orbital momentum of the smaller or x-satellite when moving at distance γ_1, but less than $4/3^{\frac{3}{4}}$ times the orbital momentum of the larger or y-satellite when moving at distance γ_2.

Third, there are maxima and minima for both sections, if the whole momentum be greater than $4/3^{\frac{3}{4}}$ times the orbital momentum of the larger or y-satellite when moving at distance γ_2.

This third case now requires further subdivision, according as whether there are not or are absolute maximum or minimum points on the surface.

If there are such points the two equations (24) or (25) must be simultaneously satisfied.

Hence we must have $n=\Omega_x=\Omega_y$, in order that there may be a maximum or minimum point on the surface.

But in this case the two satellites revolve in the same periodic time, and may be deemed to be rigidly connected together, and also rigidly connected with the planet. Hence the configurations of maximum or minimum energy are such that all three bodies move as though rigidly connected together.

The simultaneous satisfaction of (24) necessitates that

$$x^{\frac{1}{4}}=\frac{\lambda_1\kappa_2}{\kappa_1\lambda_2}y^{\frac{1}{4}} \text{ or } y^{\frac{1}{4}}=\frac{\lambda_2\kappa_1}{\kappa_2\lambda_1}x^{\frac{1}{4}}$$

Hence the equations (24) become

$$h-\left[\kappa_1+\kappa_2\left(\frac{\lambda_2\kappa_1}{\lambda_1\kappa_2}\right)^{\frac{1}{4}}\right]x^{\frac{1}{4}}=\frac{\lambda_1}{\kappa_1}\frac{1}{x^{\frac{3}{4}}}$$

$$h-\left[\kappa_1\left(\frac{\lambda_1\kappa_2}{\lambda_2\kappa_1}\right)^{\frac{1}{4}}+\kappa_2\right]y^{\frac{1}{4}}=\frac{\lambda_2}{\kappa_2}\frac{1}{y^{\frac{3}{4}}}$$

These equations may be written

$$\left.\begin{aligned}(x^{\frac{1}{4}})^4-\frac{h}{\kappa_1+\kappa_2(\lambda_2\kappa_1/\lambda_1\kappa_2)^{\frac{1}{4}}}(x^{\frac{1}{4}})^3+\frac{\lambda_1/\kappa_1}{\kappa_1+\kappa_2(\lambda_2\kappa_1/\lambda_1\kappa_2)^{\frac{1}{4}}}=0\\(y^{\frac{1}{4}})^4-\frac{h}{\kappa_1(\lambda_1\kappa_2/\lambda_2\kappa_1)^{\frac{1}{4}}+\kappa_2}(y^{\frac{1}{4}})^3+\frac{\lambda_2/\kappa_2}{\kappa_1(\lambda_1\kappa_2/\lambda_2\kappa_1)^{\frac{1}{4}}+\kappa_2}=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (27)$$

Then treating these biquadratics in the same way as before, we find that they have, or have not, two real roots, according as h is greater or less than

$$\frac{4}{3^{\frac{3}{4}}}\left[(\lambda_1\kappa_1^2)^{\frac{1}{3}}+(\lambda_2\kappa_2^2)^{\frac{1}{3}}\right]^{\frac{3}{4}}$$

Now $$(\lambda_1\kappa_1^2)^{\frac{1}{3}}+(\lambda_2\kappa_2^2)^{\frac{1}{3}}=\frac{\mu^{\frac{2}{3}}M}{C}\left[\frac{m_1}{(M+m_1)^{\frac{1}{3}}}+\frac{m_2}{(M+m_2)^{\frac{1}{3}}}\right]$$

Therefore there is, or there is not, a pair of real solutions of the equations $n=\Omega_x=\Omega_y$, according as the total momentum of the system is, or is not, greater than

$$\frac{4}{3^{\frac{3}{4}}}\,\mu^{\frac{1}{2}}C^{\frac{1}{4}}M^{\frac{3}{4}}\left[\frac{m_1}{(M+m_1)^{\frac{1}{3}}}+\frac{m_2}{(M+m_2)^{\frac{1}{3}}}\right]^{\frac{3}{4}}$$

And this is also the criterion whether or not there is a maximum, or minimum, or maximum-minimum point on the energy surface.

In the case where the masses of the satellites are small compared with the mass of the planet, we may express the critical value of the momentum of the system in the form

$$\frac{4}{3^{\frac{3}{4}}}\,\mu^{\frac{1}{2}}C^{\frac{1}{4}}\frac{[M(m_1+m_2)]^{\frac{3}{4}}}{(M+m_1+m_2)^{\frac{1}{4}}}$$

A comparison of this critical value with the two previous ones shows that if the two satellites be fused together, and if γ be such that

$$\frac{M(m_1+m_2)}{M+m_1+m_2}\gamma^2=C,$$

and if ω be the orbital angular velocity of the compound satellite when moving at distance γ, then the above critical value of the momentum of the whole system is

$$\frac{4}{3^{\frac{3}{4}}}\frac{M(m_1+m_2)}{M+m_1+m_2}\omega\gamma^2$$

and this is $4/3^{\frac{3}{4}}$ times the orbital momentum of the compound satellite when revolving at distance γ.

Hence if the masses of the satellites are small compared with that of the planet, there are, or are not, maximum or minimum or maximum-minimum points on the surface of energy, according as the total momentum of the system is greater or less than $4/3^{\frac{3}{4}}$ times the orbital momentum of the compound satellite when moving at distance γ.

In the case where the masses of the satellites are not small compared with that of the planet, I leave the criterion in its analytical form.

There are thus three critical values of the momentum of the whole system, and the actual value of the momentum determines the character of the surface of energy according to its position with reference to these critical values.

In proceeding to consider the graphical method of solution by means of the contour lines of the energy surface, I shall choose the total momentum of the system to be greater than this third critical value, and the surface will have a maximum point. From the nature of the surface in this case we shall be able to see how it would differ if the total momentum bore any other position with reference to the three critical values. It will be sufficient if we only consider the case where the masses of the two satellites are small compared with that of the planet.

By (17) we have, with an easily intelligible alternative notation,

$$\left.\begin{matrix}\kappa_1\\ \kappa_2\end{matrix}\right\}=\tfrac{5}{2}\frac{\left.\begin{matrix}m_1\\ m_2\end{matrix}\right\}}{M}\sqrt{\frac{g}{a}},\quad \left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\}=\tfrac{5}{2}\frac{\left.\begin{matrix}m_1\\ m_2\end{matrix}\right\}}{M}\frac{g}{a}$$

Now κ_1 is an angular velocity, and if we choose $1/\kappa_1$ as the unit of time, we have

$$\kappa_1=1,\ \kappa_2=\frac{m_2}{m_1}$$

also

$$\lambda_1=\tfrac{2}{5}\frac{M}{m_1}\kappa_1^2\quad \lambda_2=\tfrac{2}{5}\frac{M}{m_2}\kappa_2^2$$

Now if we choose the mass of the first satellite as unit of mass, then $m_1=1$, and we have

$$\kappa_1=1,\ \kappa_2=m_2,\ \lambda_1=\tfrac{2}{5}M,\ \lambda_2=\tfrac{2}{5}Mm_2$$

The unit of length has been already chosen as equal to the mean radius of the planet.

Then substituting in (21) we have as the equation to the energy surface

$$2z=(h-x^{\frac{1}{2}}-m_2y^{\frac{1}{2}})^2-\tfrac{2}{5}M\left(\frac{1}{x^{\frac{1}{2}}}+\frac{m_2}{y^{\frac{1}{2}}}\right)$$

And since we suppose m_1 and m_2 to be small compared with M, we have

$$x=\left(\frac{c_1}{a}\right)^{\frac{1}{2}},\ y=\left(\frac{c_2}{a}\right)^{\frac{1}{2}}$$

On account of the abruptness of the curvatures, this surface is extremely difficult to illustrate unless the figure be of very large size, and it is therefore difficult to choose appropriate values of h, M, m_2, so as to bring the figure within a moderate compass.

In order to exhibit the influence of unequal masses in the satellites, I choose $m_2=2$, the mass of the first satellite being unity. I take $M=50$, so that $\tfrac{2}{5}M=20$.

Then with these values for M and m, the first critical value for h is 3·711, the second is 6·241, and the third is 8·459.

I accordingly take $h=9$, which is greater than the third critical value. The surface to be illustrated then has the equation

$$2z=(9-x^{\frac{1}{2}}-2y^{\frac{1}{2}})^2-20\left(\frac{1}{x^{\frac{1}{2}}}+\frac{2}{y^{\frac{1}{2}}}\right)$$

There is also another surface to be considered, namely

$$n=9-x^{1}-2y^{1}$$

which gives the rotation of the planet corresponding to any values of x and y.

The equations

$$n=\Omega_x, \quad n=\Omega_y$$

have also to be exhibited.

The computations requisite for the illustration were laborious, as I had to calculate values of z and n corresponding to a large number of values of x and y, and then by graphical interpolation to find the values of x and y, corresponding to exact values of z and n.

The surface of energy will be considered first.

Plate 61 shows the contour-lines (that is to say, lines of equal energy) in the positive quadrant, z being either positive or negative.

I speak below as though the paper were held horizontally, and as though positive z were drawn vertically upwards.

The numbers written along the axes give the numerical values of x and y.

The numbers written along the curves are the corresponding values of $-2z$. Since the numbers happen to be all negative, smaller numbers indicate greater energy than larger ones; and, accordingly, in going down hill we pass from smaller to greater numbers.

The full-line contours are equidistant, and correspond to the values 9, $8\frac{1}{2}$, 8, $7\frac{1}{2}$, 7, $6\frac{1}{2}$, and 6 of $-2z$; but since the slopes of the surface are very gentle in the central part, dotted lines (. . . .) are drawn for the contours $7\frac{3}{4}$ and $7\frac{1}{4}$.

The points marked 5·529 and 7·442 are equidistant from x and y, and therefore correspond to the case when the two satellites have the same distance from the planet, or, which amounts to the same thing, are fused together. The former is a maximum point on the surface, the latter a maximum-minimum.

The dashed line (— — —) through 7·442 is the contour corresponding to that value of $-2z$.

The chain-dot lines (—·—·—) through the same point will be explained below.

An inspection of these contours shows that along the axes of x and y the surface has infinitely deep ravines; but the steepness of the cliffs diminishes as we recede from the origin.

The maximum point 5·529 is at the top of a hill bounded towards the ravines by very steep cliffs; but sloping more gradually in the other directions.

The maximum-minimum point 7·442 is on a saddle-shaped part of the surface, for we go up hill, whether proceeding towards O or away from O, and we go down hill in either direction perpendicular to the line towards O.

If the total angular momentum of the system had been less than the smallest critical value, the contour lines would all have been something like rectangular hyperbolas with the axes of x and y as asymptotes, like the outer curves marked 6, 6½, 7 in Plate 61. In this case the whole surface would have sloped towards the axes.

If the momentum had been greater than the smallest, and less than the second critical value, the outer contours would have still been like rectangular hyperbolas, and the branches which run upwards, more or less parallel to y, would still have preserved that character nearer to the axes, whilst the branches more or less parallel to x would have had a curve of contrary reflexure, somewhat like that exhibited by the curve 7½ in Plate 61, but less pronounced. In this case all the lines of steepest slope would approach the axis of x, but some of them in some part of their course would recede from the axis of y.

If the momentum had been greater than the second, but less than the third critical value, the contours would still all have been continuous curves, but for some of the inner ones there would have been contrary reflexure in both branches, somewhat like the curve marked 7½ in Plate 61. There would still have been no closed curves amongst the contours. Here some of the lines of greatest slope would in part of their course have receded from the axis of x, and some from the axis of y, but the same line of greatest slope would never have receded from both axes.

Finally, if the momentum be greater than the third critical value, we have the case exhibited in Plate 61.

Plate 62, fig. 1, exhibits the lines of greatest slope on the surface. It was constructed by making a tracing of Plate 61, and then drawing by eye the orthogonal trajectories of the contours of equal energy. The dashed line (— — —) is the contour corresponding to the maximum-minimum point 7·442 of Plate 61. The chain-dot line (—·—·—) will be explained later.

One set of lines all radiate from the maximum point 5·529 of Plate 61. The arrows on the curves indicate the downward direction. It is easy to see how these lines would have differed, had the momentum of the system had various smaller values.

Plate 62, fig. 2, exhibits the contour lines of the surface

$$n=9-x^{\frac{1}{7}}-2y^{\frac{1}{7}}$$

It is drawn on the same scale as Plate 61 and Plate 62, fig. 1.

The computations for the energy surface, together with graphical interpolation, gave values of x and y corresponding to exact values of n.

The axis of n is perpendicular to the paper, and the numbers written on the curves indicate the various values of n.

These curves are *not* asymptotic to the axes, for they all cut both axes. The angles, however, at which they cut the axes are so acute that it is impossible to exhibit the intersections.

None of the curves meet the axis of x within the limits of the figure.

The curve $n=3$ meets the axis of y when $y=2150$, and that for $n=3\frac{1}{2}$ when $y=1200$, but for values of n smaller than 3 the intersections with the axis of y do not fall within the figure. The thickness, which it is necessary to give to the lines in drawing, obviously prevents the possibility of showing these facts, except in a figure of very large size.

On the side remote from the origin of the curve marked 0, n is negative, on the nearer side positive.

Since $M=50$,

$$\Omega_x=20/x^3 \text{ and } \Omega_y=20/y^3$$

Hence the lines on the figure, for which Ω_x is constant, are parallel to the axis of y, and those for which Ω_y is constant are parallel to the axis of x.

The points are marked off along each axis for which Ω_x or Ω_y are equal to $3\frac{1}{2}$, 3, $2\frac{1}{2}$, 2, $1\frac{1}{2}$, 1. The points for which they are equal to $\frac{1}{2}$ fall outside the figure.

Now, if we draw parallels to y through these points on the axis of x, and parallels to x through the points on the axis of y, these parallels will intersect the n curves of the same magnitude in a series of points. For example, $\Omega_x=1\frac{1}{2}$, when x is about 420, and the parallel to y through this point intersects the curve $n=1\frac{1}{2}$, where y is about 740. Hence the first or x-satellite moves as a rigid body attached to the planet, when the first satellite has a distance $(420)^2$, and the second a distance $(740)^2$. In this manner we obtain a curve shown as chain-dot (—·—·—) and marked $\Omega_x=n$ for every point on which the first satellite moves as though rigidly connected with the planet; and similarly there is a second curve (—·—·—) marked $\Omega_y=n$ for every point on which the second satellite moves as though rigidly connected with the planet. This pair of curves divides space into four regions, which are marked out on the figure.

The space comprised between the two, for which Ω_x and Ω_y are both less than n, is the part which has most interest for actual planets and satellites, because the satellites of the solar system in general revolve slower than their planets rotate.

If the sun be left out of consideration, the Martian system is exemplified by the space $\Omega_x>n$, $\Omega_y<n$, because the smaller and inner satellite revolves quicker than the planet rotates, and the larger and outer one revolves slower.

The little quadrilateral space near O is of the same character as the external space $\Omega_x>n$, $\Omega_y>n$, but there is not room to write this on the figure.

These chain-dot curves are marked also on Plates 61 and 62, fig. 1. In Plate 61 the line $\Omega_x=n$ passes through all those points on the contours of energy whose tangents are parallel to x, and the line $\Omega_y=n$ passes through points whose tangents are parallel to y.

The tangents to the lines of greatest slope are perpendicular to the tangents to the contours of energy; hence in Plate 62, fig. 1, $\Omega_x=n$ passes through points whose tangents are parallel to y, and $\Omega_y=n$ through points whose tangents are parallel to x.

Within each of the four regions into which space is thus divided the lines of slope preserve the same character; so that if, for example, at any part of the region they are receding from x and y, they do so throughout.

This is correct, because dx/dt changes sign with $n-\Omega_x$ and dy/dt with $n-\Omega_y$; also either $n-\Omega_x$ or $n-\Omega_y$ changes sign in passing from one region to another. In these figures a line drawn at 45° to the axes through the origin divides the space into two parts; in the upper region y is greater than x, and in the lower x is greater than y. Hence configurations, for which the greater or y-satellite is exterior to the lesser or x-satellite, are represented by points in the upper space and those in which the lesser satellite is exterior by the lower space.

In the figures of which I have been speaking hitherto the abscissas and ordinates are the $\frac{7}{2}$ power of the distances of the two satellites; now this is an inconveniently high power, and it is not very easy to understand the physical meaning of the result. I have therefore prepared another figure in which the abscissas and ordinates are the actual distances. In Plate 63, fig. 3, the curves are no longer lines of steepest slope.

The reduction from Plate 62, fig. 1, to Plate 63, fig. 3, involved the raising of all the ordinates and abscissas of the former one to the $\frac{2}{7}$ power. This process was rather troublesome, and Plate 63, fig. 3, cannot claim to be drawn with rigorous accuracy; it is, however, sufficiently exact for the hypothetical case under consideration. If we had to treat any actual case, it would only be necessary to travel along a single line of change, and for that purpose special methods of approximation might be found for giving more accurate results.

In this figure the numbers written along the axes denote the distances of the satellites in mean radii of the planet—the radius of the planet having been chosen as the unit of length.

The chain-dot curves, as before, enclose the region for which the orbital angular velocities of the satellites are less than that of the planet's rotation. The line at 45° to the axes marks out the regions for which the larger satellite is exterior or interior to the smaller one.

Let us consider the closed space, within which Ω_x and Ω_y are less than n.

The corner of this space is the point of maximum energy, from which all the curves radiate.

Those curves which have tangents inclined at more than 45° to the axis of x denote that, during part of the changes, the larger satellite recedes more rapidly from the planet than the smaller one.

If the curve cuts the 45° line, it means that the larger satellite catches up the smaller one. Since these curves all pass from the lower to the upper part of the space, it follows that this will only take place when the larger satellite is initially interior. According to the figure, after catching up the smaller satellite, the larger satellite becomes exterior. In reality there would probably either be a collision or the pair of satellites would form a double system like the earth and moon. After this the smaller satellite becomes almost stationary, revolves for an instant as though rigidly connected with the planet, and then slower than the planet revolves (when the curve passes out of the closed space); the smaller satellite then falls into the planet, whilst the larger satellite maintains a sensibly constant distance from the planet.

If we take one of the other curves corresponding to the case of the larger satellite being interior, we see that the smaller satellite may at first recede more rapidly than the larger, and then the larger more rapidly than the smaller, but not so as to catch it up. The larger one then becomes nearly stationary, whilst the smaller one still recedes. The larger one then falls in, whilst the smaller one is nearly stationary.

If we now consider those curves which are from the beginning in the upper half of the closed space, we see that if the larger satellite is initially exterior, it recedes at first rapidly, whilst the smaller one recedes slowly. The smaller and inner satellite then comes to revolve as though rigidly connected with the planet, and afterwards falls into the planet, whilst the distance of the larger one remains nearly unaltered.

Either satellite comes into collision with the planet when its distance therefrom is unity. When this takes place the colliding satellite becomes fused with the planet, and the system becomes one where there is only a single satellite; this case might then be treated as in previous papers.

The divergences of the curves from the point of maximum energy shows that a very small difference of initial configuration in a pair of satellites may in time lead to very wide differences of configuration. Accordingly tidal friction alone will not tend to arrange satellites in any determinate order. It cannot, therefore, be definitely asserted that tidal friction has not operated to arrange satellites in any order which may be observed.

I have hitherto only considered the positive quadrant of the energy surface, in which both satellites revolve positively about the planet. There are, however, three other cases, viz.: where both revolve negatively (in which case the planet necessarily revolves positively, so as to make up the positive angular momentum), or where one revolves negatively and the other positively.

These cases will not be discussed at length, since they do not possess much interest.

Plate 63, fig. 4, exhibits the contours of energy for that quadrant in which the smaller or x-satellite revolves positively and the larger or y-satellite negatively. This figure may be conceived as joined on to Plate 61, so that the x-axes coincide. The numbers written on the contours are the values of $2z$; they are positive and pretty large. Whence it follows that these contours are enormously higher than those shown in Plate 61, where all the numbers on the contours were negative.

The contours explain the nature of the surface. It may, however, be well to remark that, although the contours appear to recede back from the x-axis for ever, this is not the case; for, after receding from the axis for a long way, they ultimately approach it again, and the axis is asymptotic to each of them. The point, at which the tangent to each contour is parallel to the axis of x, becomes more and more remote the higher the contour.

The lines of steepest slope on this surface give, as before, the solution of the problem.

If we hold this figure upside down, and read x for y and y for x, we get a figure which represents the general nature of the surface for the case where the x-satellite revolves negatively and the y-satellite positively. But of course the figure would not be drawn correctly to scale.

The contours for the remaining quadrant, in which both satellites revolve negatively would somewhat resemble a family of rectangular hyperbolas with the axes as asymptotes. I have not thought it worth while to construct them, but the physical interpretation is obviously that both satellites always must approach the planet.

II.

A DISCUSSION OF THE EFFECTS OF TIDAL FRICTION WITH REFERENCE TO THE EVOLUTION OF THE SOLAR SYSTEM.

§ 6. *General consideration of the problem presented by the solar system.*

In a series of previous papers I have traced out the changes in the manner of motion of the earth and moon which must have been caused by tidal friction. By adopting the hypothesis that tidal friction has been the most important element in the history of those bodies, we are led to coordinate together all the elements in their motions in a manner so remarkable, that the conclusion can hardly be avoided that the hypothesis contains a great amount of truth.

Under these circumstances it is natural to inquire whether the same agency may not have been equally important in the evolution of the other planetary sub-systems, and of the solar system as a whole.

This inquiry necessarily leads on to wide speculations, but I shall endeavour to derive as much guidance as possible from numerical data.

In the first part of the present paper the theory of the tidal friction of a planet, attended by several satellites, has been treated.

It would, at first sight, seem natural to replace this planet by the sun, and the satellites by the planets, and to obtain an approximate numerical solution. We might suppose that such a solution would afford indications as to whether tidal friction has or has not been a largely efficient cause in modifying the solar system.

The problem here suggested for solution differs, however, in certain points from that actually presented by the solar system, and it will now be shown that these differences are such as would render the solution of no avail.

The planets are not particles, as the suggested problem would suppose them to be, but they are rotating spheroids in which tides are being raised both by their own satellites and by the sun. They are, therefore, subject to a complicated tidal friction; the reaction of the tides raised by the satellites goes to expand the orbits of the satellites, but the reaction of the tide raised in the planet by the sun, and that raised in the sun by the planet both go towards expanding the orbit of the planet. It is this latter effect with which we are at present concerned.

I propose then to consider the probable relative importance of these two causes of change in the planetary orbits.

But before doing so it will be well as a preliminary to consider another point.

In considering the effects of tidal friction the theory has been throughout adopted that the tidally-disturbed body is homogeneous and viscous. Now we know that the planets are not homogeneous, and it seems not improbable that the tidally-disturbed parts will be principally more or less superficial—as indeed we know that they are in the case of terrestrial oceans. The question then arises as to the extent of error introduced by the hypothesis of homogeneity.

For a homogeneous viscous planet we have shown that the tidal frictional couple is approximately equal [*] to

$$C\frac{\tau^2}{\mathfrak{g}}\,\frac{n-\Omega}{\mathfrak{p}},\ \text{where } \mathfrak{p}=\frac{gaw}{19v}$$

Now how will this expression be modified, if the tidally-disturbed parts are more or less superficial, and of less than the mean density of the planet?

To answer this query we must refer back to the mannér in which the expression was built up.

By reference to my paper "On the Tides of a Viscous Spheroid" (Phil. Trans., Part I., 1879, pp. 8–10, especially the middle of p. 8), it will be seen that $\mathfrak{p}$ is really $(\frac{5}{2}gaw-\frac{3}{2}gaw)/19v$, and that in both of these terms w represents the density of the tidally-disturbed matter, but that in the former g represents the gravitation of the planet and in the second it is equal to $\frac{4}{3}\pi\mu aw$, where w is the density of the tidally-disturbed matter. Now let f be the ratio of the mean density of the spheroid to the density of the tidally-disturbed matter.

Then in the former term

$$gaw=\frac{3}{4\pi\mu}\cdot g\cdot\tfrac{4}{3}\pi\mu aw=\frac{3}{4\pi\mu}g^2\times\frac{1}{f}$$

And in the latter

$$gaw=\frac{3}{4\pi\mu}g^2\times\frac{1}{f^2}$$

Hence if the planet be heterogeneous and the tidally-disturbed matter superficial, $\mathfrak{p}$ must be a coefficient of the form

$$\frac{3}{4\pi\mu\times 19}\,\frac{g^2}{vf}\left(\frac{5}{2}-\frac{3}{2f}\right)$$

If f be unity this reduces to the form $gaw/19v$, as it ought; but if the tidally-disturbed matter be superficial and of less than the mean density, then $\mathfrak{p}$ must be a coefficient which varies as $\frac{g^2}{\mu v f}\left(1-\frac{3}{5f}\right)$. The exact form of the coefficient will of course depend upon the exact nature of the tides. If f be large the term $3/5f$ will be negligeable compared with unity. Again, if we refer to the following paper ("Precession, &c.," Phil. Trans., Part II., 1879, p. 456), it will be seen that the C in the expression for the tidal frictional couple represents $\frac{2}{5}(\frac{4}{3}\pi a^3 w)a^2$, where w is the density of the tidally-disturbed matter; hence C should be replaced by C/f.

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[*] I leave out of account the case of "large" viscosity, because as shown in a previous paper that could only be true of a planet which in ordinary parlance would be called a solid of great rigidity.—See "Precession," Phil. Trans., 1879, Part II., p. 531.

Then if we reconstruct the expression for the tidal frictional couple, we see that it is to be divided by f, because of the true meaning to be assigned to C, but is to be multiplied by f on account of the true meaning to be assigned to $\mathfrak{p}$.

From this it follows that for a given viscosity it is, roughly speaking, probable that the tidal frictional couple will be nearly the same as though the planet were homogeneous. The above has been stated in an analytical form, but in physical language the reason is because the lagging of the tide will be augmented by the deficiency of density of the tidally-disturbed matter in about the same proportion as the frictional couple is diminished by the deficiency of density of the tide-wave upon which the disturbing satellite has to act.

This discussion appeared necessary in order to show that the tidal frictional couple is of the same order of magnitude whether the planet be homogeneous or heterogeneous, and that we shall not be led into grave errors by discussing the theory of tidal friction on the hypothesis of the homogeneity of the tidally-disturbed bodies.

We may now proceed to consider the double tidal action of a planet and the sun.

Let us consider the particular homogeneous planet whose mass, distance from sun, and orbital angular velocity are m, c, Ω. For this planet, let $C'=$ moment of inertia; $a'=$ mean radius; $w'=$ density; $g'=$ gravity; $\mathfrak{g}'=\frac{2}{5}g'/a'$; $v'=$ the viscosity; $\mathfrak{p}'=g'a'w'/19v'=\frac{3}{4\times 19}g'^2/\pi\mu v'$; and $n'=$ angular velocity of diurnal rotation.

The same symbols when unaccented are to represent the parallel quantities for the sun.

Now suppose the sun to be either perfectly rigid, perfectly elastic, or perfectly fluid. Then *mutatis mutandis*, equation (2) gives the rate of increase of the planet's distance from the sun under the influence of the tidal friction in the planet. It becomes

$$\mu^{\frac{1}{2}}\frac{Mm}{(M+m)^{\frac{1}{2}}}\frac{dc^{\frac{1}{2}}}{dt}=\left(\tfrac{3}{2}\right)^2\frac{C'}{\mathfrak{g}'}\frac{(\mu M)^2}{c^6}\frac{n'-\Omega}{\mathfrak{p}'}$$

If the planet have no satellite the right-hand side is equal to $-C'dn'/dt$, because the equation was formed from the expression for the tidal frictional couple.

Hence, if none of the planets had satellites we should have a series of equations of the form

$$C'n'+\mu^{\frac{1}{2}}\frac{Mm}{(M+m)^{\frac{1}{2}}}c^{\frac{1}{2}}=h$$

with different h's corresponding to each planet.

We may here remark that the secular effects of tidal friction in the case of a rigid sun attended by tidally-disturbed planets, with no satellites, may easily be determined. For if we put $c^{\frac{1}{2}}=x$, and note that Ω varies as x^{-3}, and that n' has the form $(h-kx)/C'$, we see that it would only be necessary to evaluate a series of integrals of the form $\int_{x_0}^{x}\frac{\epsilon x^{15}dx}{\alpha-\beta x^3+\gamma x^4}$. This integral is in fact merely the time which elapses whilst x changes from x_0 to x, and the time scale is the same for all the planets. It is not at present worth while to pursue this hypothetical case further.

Now if we suppose the planet to raise frictional tides in the sun, as well as the sun to raise tides in the planets, we easily see by a double application of (2) that

$$\mu^{\frac{1}{2}}\frac{Mm}{(M+m)^{\frac{1}{2}}}\frac{dc^{\frac{1}{2}}}{dt}=\left(\tfrac{3}{2}\right)^2\frac{1}{c^6}\left[(\mu m)^2\frac{C}{\mathfrak{g}\mathfrak{p}}(n-\Omega)+(\mu M)^2\frac{C'}{\mathfrak{g}'\mathfrak{p}'}(n'-\Omega)\right] \quad . \quad . \quad . \quad (28)$$

The tides raised in the planet by its satellites do not occur explicitly in this equation, but they do occur implicitly, because n', the planet's rotation, is affected by these tides.

The question which we now have to ask is whether in the equation (28) the solar term (without accents) or the planetary term (with accents) is the more important.

In the solar system the rotations of the sun and planets are rapid compared with the orbital motions, so that Ω may be neglected compared with both n and n'.

Hence the planetary term bears to the solar term approximately the ratio $\frac{M^2C''n'\mathfrak{g}\mathfrak{p}}{m^2Cn\mathfrak{g}'\mathfrak{p}'}$.

Now $\left(\frac{M}{m}\right)^2\frac{C'}{C}\frac{\mathfrak{g}}{\mathfrak{g}'}=\frac{M}{m}\left(\frac{a'}{a}\right)^2\frac{g}{g'}\frac{a'}{a}=\left(\frac{g}{g'}\right)^2\frac{a'}{a}$. Also $\frac{\mathfrak{p}}{\mathfrak{p}'}=\left(\frac{g}{g'}\right)^2\frac{v'}{v}$.

Therefore the ratio is $\left(\frac{g}{g'}\right)^4\frac{a'}{a}\frac{n'}{n}\frac{v'}{v}$

Now solar gravity is about 26·4 times that of the earth and about 10·4 times that of Jupiter. The solar radius is about 109 times that of the earth and about 10 times that of Jupiter. The earth's rotation is about 25·4 times that of the sun, and Jupiter's rotation is about 61 times that of the sun. Combining these data I find that the effect of solar tides in the earth is about 113,000 v'/v times as great as the effect of terrestrial tides in the sun, and the effect of solar tides in Jupiter is about 70,000 v'/v times as great as the effect of Jovian tides in the sun. It is not worth while to make a similar comparison for any of the other planets.

Now it seems reasonable to suppose that the coefficient of tidal friction in the planets is of the same order of magnitude as in the sun, so that it is improbable that v'/v should be either a large number or a small fraction.

We may conclude then from this comparison that the effects of tides raised in the sun by the planets are quite insignificant in comparison with those of tides raised in the planets by the sun.

It appears therefore that we may fairly leave out of account the tides raised in the sun in studying the possible changes in the planetary orbits as resulting from tidal friction.

But the difference of physical condition in the several planets is probably considerable, and this would lead to differences in the coefficients of tidal friction to which there is no apparent means of approximating. It therefore seems inexpedient at present to devote time to the numerical solution of the problem of the rigid sun and the tidally-disturbed planets.

§ 7. *Numerical data and deductions therefrom.*

Although we are thus brought to admit that it is difficult to construct any problem which shall adequately represent the actual case, yet a discussion of certain numerical values involved in the solar system and in the planetary subsystems will, I think, lead to some interesting results.

The fundamental fact with regard to the theory of tidal friction is the transformation of the rotational momentum of the planet as it is destroyed by tidal friction into orbital momentum of the tide-raising body.

Hence we may derive information concerning the effects of tidal friction by the evaluation of the various momenta of the several parts of the solar system.

Professor J. C. ADAMS has kindly given me a table of values of the planetary masses, each with its attendant satellites. The authorities were as follows: for Mercury, ENCKE; for Venus, LE VERRIER; for the Earth, HANSEN; for Mars, HALL; for Jupiter, BESSEL; for Saturn, BESSEL; for Uranus, VON ASTEN; for Neptune, NEWCOMB.

The masses were expressed as fractions of the sun. The results, when earth plus moon is taken as unity, are given in the table below. The mean distances, taken from HERSCHEL's 'Astronomy,' are given in a second column.

	Masses (m).	Mean Distances (c.).
Sun . . .	315,511·	
Mercury . .	·06484	·387098
Venus . .	·78829	·723332
Earth . .	1·00000	1·000000
Mars . . .	·10199	1·523692
Jupiter . .	301·0971	5·202776
Saturn . .	90·1048	9·538786
Uranus . .	14·3414	19·18239
Neptune. .	16·0158	30·05660

The unit of mass is earth plus moon, the unit of length is the earth's mean distance from the sun, and the unit of time will be taken as the mean solar day.

Then μ being the attraction between unit masses at unit distance, M being sun's mass, and 365·25 being the earth's periodic time, we have

$$\sqrt{\mu M}=\frac{2\pi}{365{\cdot}25}=\frac{10^{8\cdot23558}}{10^{10}}$$

The momentum of orbital motion of any one of the planets round the sun is given by $m.\sqrt{\mu M}.\sqrt{c}$.

With the above data I find the following results.*

TABLE I.

Planet	Orbital momentum.
Mercury	·00079
Venus	·01309
Earth	·01720
Mars	·00253
Jupiter	13·469
Saturn	5·456
Uranus	1·323
Neptune	1·806
Total	22·088

We must now make an estimate of the rotational momentum of the sun, so as to compare it with the total orbital momentum of the planets.

It seems probable that the sun is much more dense in the central portion, than near the surface.† Now if the Laplacian law of internal density were to hold with the sun, but with the surface density infinitely small compared with the mean density, we should have

$$C=\tfrac{2}{3}\left[1-\frac{6}{\pi^2}\right]Ma^2$$

If on the other hand the sun were of uniform density we should have $C=\frac{2}{5}Ma^2$‡.

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* These values are of course not rigorously accurate, because the attraction of Jupiter and Saturn on the internal planets is equivalent to a diminution of the sun's mass for them, and the attraction of the internal planets on the external ones is equivalent to an increase of the sun's mass.

† I have elsewhere shown that there is a strong probability that this is the case with Jupiter, and that planet probably resembles the sun more nearly than does the earth.—See Ast. Soc Month. Not., Dec., 1876.

‡ These considerations lead me to remark that in previous papers, where the tidal theory was applied numerically to the case of the earth and moon, I might have chosen more satisfactory numerical values with which to begin the computations.

It was desirable to use a consistent theory of frictional tides, and that founded on the hypothesis of a homogeneous viscous planet was adopted.

The earth had therefore to be treated as homogeneous, and since tidal friction depends on relative

The former of these two suppositions seems more likely to be near the truth than the latter.

Now $\frac{2}{3}(1-6\pi^{-2})=\cdot26138$, so that C may lie between $(\cdot26138)Ma^2$ and $(\cdot4)Ma^2$.

The sun's apparent radius is 961″·82, therefore the unit of distance being the present distance of the earth from the sun, $a=961\cdot82\pi/648{,}000$; also $M=315{,}511$.

Lastly the sun's period of rotation is about 25·38 m.s. days, so that $n=2\pi/25\cdot38$.

Combining these numerical values I find that Cn (the solar rotational momentum) may lie between ·444 and ·679. The former of these values seems however likely to be far nearer the truth than the latter.

It follows therefore that the total orbital momentum of the planetary system, found above to be 22, is about 50 times that of the solar rotation.

motion, the rotation of the homogeneous planet had to be made identical with that of the real earth. A consequence of this is that the rotational momentum of the earth in my problem bore a larger ratio to the orbital momentum of the moon than is the case in reality. Since the consequence of tidal friction is to transfer momentum from one part of the system to the other, this treatment somewhat vitiated subsequent results, although not to such an extent as could make any important difference in a speculative investigation of that kind.

If it had occurred to me, however, it would have been just as easy to have replaced the actual heterogeneous earth by a homogeneous planet mechanically equivalent thereto. The mechanical equivalence referred to lies in the identity of mass, moment of inertia, and rotation between the homogeneous substitute and the real earth. These identities of course involve identity of rotational momentum and of rotational energy, and, as will be seen presently, other identities are approximately satisfied at the same time.

Suppose that roman letters apply to the real earth and italic letters to the homogeneous substitute.

By LAPLACE's theory of the earth's figure, with THOMSON and TAIT's notation ('Nat. Phil.,' § 824)

$$\mathrm{C}=\tfrac{2}{3}\left\{1-\frac{6(f-1)}{f\theta^2}\right\}\mathrm{Ma}^2$$

where f is the ratio of mean to surface density, and θ is a certain angle.

Also

$$C=\tfrac{2}{5}Ma^2\left(1+\frac{2e}{3}\right)$$

where e is the ellipticity of the homogeneous planet's figure.

Then by the above conditions of mechanical identity

$$\mathrm{M}=M \text{ and } \mathrm{C}=C$$

whence

$$\left(\frac{a}{\mathrm{a}}\right)^2=\tfrac{5}{3}\left(1-\frac{2e}{3}\right)\left\{1-\frac{6(f-1)}{f\theta^2}\right\}$$

Now put $\mathrm{m}=\mathrm{n}^2\mathrm{a}/\mathrm{g}$, $m=n^2a/g$; where g, g are mean pure gravity in the two cases. Then the remaining condition gives $\mathrm{n}=n$.

Therefore

$$\frac{m}{\mathrm{m}}=\frac{\mathrm{g}a}{g\mathrm{a}}=\left(\frac{a}{\mathrm{a}}\right)^3$$

But

$$e=\tfrac{5}{4}m=\tfrac{5}{4}\mathrm{m}\left(\frac{a}{\mathrm{a}}\right)^3$$

Hence

$$\left(\frac{a}{\mathrm{a}}\right)^2=\tfrac{5}{3}\left\{1-\tfrac{5}{6}\mathrm{m}\left(\frac{a}{\mathrm{a}}\right)^3\right\}\left\{1-\frac{6(f-1)}{f\theta^2}\right\}$$

This is an equation which gives the radius of the homogeneous substituted planet in terms of that of

In discussing the various planetary subsystems I take most of the numerical values from the excellent tables of astronomical constants in Professor BALL's 'Astronomy,'* and from the table of masses given above.

Mercury.

The diameter at distance unity is about 6″·5; the diurnal period is 24h 0m 50s (?) The value of the mass seems very uncertain, but I take ENCKE's value given above. Assuming that the law of internal density is the same as in the earth (see below), we have $C = \cdot 33438 ma^2$, and $n = 2\pi$ very nearly. Whence I find for the rotational momentum

$$Cn = \frac{\cdot 34}{10^{10}}.$$

Venus.

The diameter at distance unity is about 16″·9; the diurnal period is 23h 21m 22s (?). Assuming the same law of internal density as for the earth, I find

$$Cn = \frac{28\cdot 6}{10^{10}}.$$

HERSCHEL remarks ('Outlines of Astronomy,' § 509) that "both Mercury and Venus have been concluded to revolve on their axes in about the same time as the Earth, though in the case of Venus, BIANCHINI and other more recent observers have contended

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the earth. It may be solved approximately by first neglecting $\frac{5}{6} \mathrm{m}(a/\mathrm{a})^3$, and afterwards using the approximate value of a/a for determining that quantity.

The density of the homogeneous planet is found from

$$w = \mathrm{w}\left(\frac{\mathrm{a}}{a}\right)^3$$

where w is the earth's mean density.

To apply these considerations to the earth, we take $\theta = 142° \, 30'$, $f = 2\cdot 057$, which give $\frac{1}{295}$ as the ellipticity of the earth's surface.

Then with these values (THOMSON and TAIT's 'Nat. Phil.,' § 824, table, col. vii., they give however ·835)

$$\tfrac{5}{8}\left\{1 - \frac{6(f-1)}{f\theta^2}\right\} = \cdot 83595$$

The first approximation gives $\frac{a}{\mathrm{a}} = \cdot 9143$, and the second $\frac{a}{\mathrm{a}} = \cdot 9133$.

Hence the radius of the actual earth 6,370,000 meters becomes, in the homogeneous substitute, 5,817,000 meters.

Taking 5·67 as the earth's mean specific gravity, that of the homogeneous planet is 7·44.

The ellipticity of the homogeneous planet is ·00329 or $\frac{1}{303}$, which differs but little from that of the real earth, viz.: $\frac{1}{295}$.

The precessional constant of the homogeneous planet is equal to the ellipticity, and is therefore ·00329. If this be compared with the precessional constant ·00327 of the earth, we see that the homogeneous substitute has sensibly the same precession as has the earth.

If a similar treatment be applied to Jupiter, then (with the numerical values given in a previous paper, Ast. Soc. Month. Not., Dec. 1876) the homogeneous planet has a radius equal to ·8 of the actual one; its density is about half that of the earth, and its ellipticity is $\frac{1}{19}$.

* 'Text Book of Science: Elements of Astronomy.' LONGMANS. 1880.

for a period of twenty-four times that length." He evidently places little reliance on the observations.

The Earth.

I adopt LAPLACE'S theory of internal density (with THOMSON and TAIT'S notation), and take, according to Colonel CLARKE, the ellipticity of surface to be $\frac{1}{295}$. This value corresponds with the value 2·057 for the ratio of mean to surface density (the f of THOMSON and TAIT), and to 142° 30′ for the auxiliary angle θ.

The moment of inertia is given by the formula

$$C=\tfrac{2}{3}\left[1-\frac{6(f-1)}{f\theta^2}\right]ma^2.$$

These values of θ and f give $\tfrac{5}{3}\left[1-\frac{6(f-1)}{f\theta^2}\right]=\cdot 83595.$

Whence $$C=\cdot 33438ma^2.$$

The numerical coefficient is the same as that already used in the case of the two previous planets.

The moon's mass being $\frac{1}{82}$nd of the earth's, the earth's mass is $\frac{82}{83}$ in the chosen unit of mass.

With sun's parallax 8″·8, and unit of length equal to earth's mean distance

$$a=\frac{8\cdot 8\pi}{648000}$$

The angular velocity of diurnal rotation, with unit of time equal to the mean solar day,

$$n=\frac{2\pi}{\cdot 99727}$$

Combining these values I find for the earth's rotational momentum

$$Cn=\frac{37\cdot 88}{10^{10}}$$

Writing m' for the moon's mass, and neglecting the eccentricity of the lunar orbit, the moon's orbital momentum* is

$$\frac{mm'}{m+m'}\Omega c^2$$

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* If we determine μ from the formula

$$\mu=\Omega^2c^3=\left(\frac{2\pi}{27\cdot 3217}\right)^2\left(\frac{8\cdot 8}{3422\cdot 3}\right)^3$$

and observe that $\mu M=(2\pi/365\cdot 25)^2$, we obtain 329,000 as the sun's mass. This disagrees with the value 315,511 used elsewhere. The discrepancy arises from the neglect of solar perturbation of the moon, and of planetary perturbation of the earth.

Taking the moon's parallax as $3422''\cdot 3$ (which gives a distance of 60·27 earth's radii), and the sun's parallax as $8''\cdot 8$, we have

$$c=\frac{8\cdot 8}{3422\cdot 3}$$

Taking the lunar period as 27·3217 m.s. days we have

$$\Omega=\frac{2\pi}{27\ 3217}$$

As above stated, m is $\frac{82}{83}$, and m' is $\frac{1}{83}$; whence it will be found that the moon's orbital momentum is $\frac{181}{10^{10}}$.

This is 4·78 times the earth's rotational momentum.

The resultant angular momentum of the system, with obliquity of ecliptic 23° 28′, is 5·71 times the earth's rotational momentum, and is $\frac{216}{10^{10}}$.

Mars.

The polar diameter at distance unity is $9''\cdot 352$ (HARTWIG, 'Nature,' June 3, 1880). With an ellipticity $\frac{1}{300}$ this gives $4''\cdot 686$ as the mean radius. The diurnal period is $24^{\rm h}\ 37^{\rm m}\ 23^{\rm s}$. Assuming the law of internal density to be the same as in the earth I find

$$Cn=\frac{1\cdot 08}{10^{10}}$$

The masses of the satellites are very small, and their orbital momentum must also be very small.

Jupiter.

The polar and equatorial diameters at the planet's mean distance from the sun are $35''\cdot 170$ and $37''\cdot 563$ (KAISER and BESSEL, 'Ast. Nach,' vol. 48, p. 111). These values give a mean radius $5\cdot 2028\times 18''\cdot 383$ at distance unity.

The period of rotation is $9^{\rm h}\ 55\frac{1}{2}^{\rm m}$, or ·4136 m.s. day.

I have elsewhere shown reason to believe that the surface density of Jupiter is very small compared with the mean density. It appears that we have approximately

$$C=\tfrac{2}{3}\times\frac{1}{2\cdot 528}ma^2=\cdot 2637\ ma^2.^*$$

The numerical coefficient differs but little from that which we should have, if the Laplacian law of internal density were true, with infinitely small surface density (f infinite, $\theta=180^\circ$); for, as appeared in considering the sun's moment of inertia, the factor would be in that case ·26138.

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[*] Ast. Soc. Month. Not., Dec., 1876, p. 83.

With these values I find

$$Cn=\frac{2{,}594{,}000}{10^{10}}={\cdot}0002594.$$

The distances of the satellites referred to the mean distance of Jupiter from the sun are

I.	II.	III.	IV.
111″·74	177″·80	283″·61	498″·87

Then taking Jupiter's mean distance to be 5·20278, the logarithms of the distances in terms of the earth's distance from the sun are

I.	II.	III.	IV.
7·45002—10	7·65174—10	7·85453—10	8·09980—10

The periodic times are in m.s. days (HERSCHEL's 'Astronomy,' Appendix)

I.	II.	III.	IV.
1·76914	3·55181	7·15455	16·6888

The masses given me by Professor ADAMS[*] from a revision of DAMOISEAU's work are in terms of Jupiter's mass.

I.	II.	III.	IV.
$\frac{2{\cdot}8311}{10^5}$	$\frac{2{\cdot}3236}{10^5}$	$\frac{8{\cdot}1245}{10^5}$	$\frac{2{\cdot}1488}{10^5}$

Combining these data according to the formula $m\Omega c^2$, where m is the mass of the satellite, I find for the orbital momenta of the satellites expressed in terms of the chosen units—

I.	II.	III.	IV.
$\frac{2406}{10^{10}}$	$\frac{2489}{10^{10}}$	$\frac{10993}{10^{10}}$	$\frac{3857}{10^{10}}$

The sum of these is $19745/10^{10}$ and is the total orbital momentum of the satellites. It is $\frac{1}{130}$th of the rotational momentum of the planet as found above.

The whole angular momentum of the Jovian system is $\frac{2{,}614{,}000}{10^{10}}$.

Saturn.

There seems to be much doubt as to the diameter of the planet.

The values of the *mean* radius at distance unity given by BESSEL, DE LA RUE, and

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[*] He kindly gave me these data for another purpose.—See Ast. Soc. Month. Not., Dec., 1876, p. 81.

MAIN (with ellipticities 1/10·2, 1/11(?), 1/9·227 respectively) are 79″, 82″, and 94″ respectively.*

The period of rotation is $10^h\ 29\frac{1}{4}^m$ or ·437 m.s. day.

Assuming (as with the sun) that the surface density is infinitely small compared with the mean density, we have $C = \cdot 2614ma^2$. I find then that these three values give respectively,

$$Cn = \frac{497{,}000}{10^{10}} \text{ and } \frac{535{,}000}{10^{10}} \text{ and } \frac{703{,}000}{10^{10}}$$

The masses of the satellites are unknown, but HERSCHEL thinks that Titan is nearly as large as Mercury.

If we take its mass as ·06 in terms of the earth's mass, its distance as 176″·755 at the planet's mean distance from the sun, and its periodic time as 15·95 m.s. days, we find the orbital momentum to be $16{,}000/10^{10}$. The whole orbital momentum of the satellites and the ring is likely to be greater than this, for the ring has been variously estimated to have a mass equal to $\frac{1}{120}$th to $\frac{1}{620}$th of the planet.

It is probable therefore that orbital momentum of the system is $\frac{1}{30}$th, or thereabouts, of the rotational momentum of the planet.

Nothing is known concerning the rotation of Uranus and Neptune, and but little of their satellites.

The results of this numerical survey of the planets are collected in the following table.

TABLE II.

Planet.	i. Rotational momentum of planet $\times 10^{10}$.	ii. Orbital momentum of satellites $\times 10^{10}$.	iii. Ratio of ii. to i.	iv. Total momentum of each planet's system $\times 10^{10}$.
Mercury . . .	·34 ?	. .	. .	·34 ?
Venus . . .	28·6 ?	. .	. .	28·6 ?
Earth. . . .	37·88	181	4·78	216
Mars	1·08	very small	very small	1·08
Jupiter . . .	2,594,000	20,000	$\frac{1}{130}$	2,614,000
Saturn . . .	500,000 to 700,000	16,000 or more	$\frac{1}{30}$ or more	520,000 to 720,000

The numbers marked with queries are open to much doubt.

If the numbers given in column iv. of this table be compared with those given in Table I., it will be seen that the total internal momentum of each of the planetary subsystems is very small compared with the orbital momentum of the planet in its motion round the sun. This ratio is largest in the case of Jupiter, and here the internal momentum is ·00026 whilst the orbital momentum is 13; hence in the case of Jupiter the orbital momentum is 5000 times the sum of the rotational momentum of the planet and the orbital momentum of its satellites. From this it follows that if the whole of the momentum of Jupiter and his satellites were destroyed by solar tidal friction, the mean distance of Jupiter from the sun would only be increased by $\frac{1}{2500}$th part. The effect of the destruction of the internal momentum of any of the other planets would be very much less.

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* Deduced from values of the equatorial diameter found by these observers, referred to the planet's mean distance from the sun, as given by BALL.

If therefore the orbits of the planets round the sun have been considerably enlarged, during the evolution of the system, by the friction of the tides raised in the planets by the sun, the primitive rotational momentum of the planetary bodies must have been thousands of times greater than at present. If this were the case then the enlargement of the orbits must simultaneously have been somewhat increased by the reaction of the tides raised in the sun by the planets.

But it does not seem probable that the planetary masses ever possessed such an enormous amount of rotational momentum, and therefore it is not probable that tidal friction has considerably affected the dimensions of the planetary orbits.

It is difficult to estimate the degree of attention which should be paid to BODE's empirical law concerning the mean distances of the planets, but it may perhaps be supposed that that law (although violated in the case of Neptune, and only partially satisfied by the asteroids) is the outcome of the laws governing the successive epochs of instability in the history of a rotating and contracting nebula. Now if, after the genesis of the planets, tidal friction had considerably affected the planetary distances, then all appearance of such primitive law in the distances would be thereby obliterated. If therefore there be now observable a sort of law of mean distances, it to some extent falls in with the conclusion arrived at by the preceding numerical comparisons.

The extreme relative smallness of the masses of the Martian and Jovian satellites tends to show the improbability of very large changes in the dimensions of the orbits of those satellites; although the argument has not nearly equal force in these cases, because the distances of the satellites from these planets is small.

The numbers given in column iii. of Table II. show in a striking manner the great difference between the present physical conditions of the terrestrial system and those of Mars, Jupiter, and Saturn. These numbers may perhaps be taken as representing the amount of effect which the tidal friction due to the satellites has had in their evolution, and confirms the conclusion that, whilst tidal friction may have been (and according to previous investigations certainly appears to have been) the great factor in the evolution of the earth and moon, yet with the satellites of the other planets it has not had such important effects.

In previous papers the expansion of the lunar orbit under the influence of terrestrial tidal friction was examined, and the moon was traced back to an origin close to the present surface of the earth. The preceding numerical comparisons suggest that the contraction of the planetary masses has elsewhere been the more important factor, and that the genesis of satellites occurred elsewhere earlier in the evolution.

It has been shown that the case of the earth and moon does actually differ widely from that of the other planets, and we may therefore reasonably suppose that the history has also differed considerably.

Although we might perhaps leave the subject at this point, yet, after arriving at the above conclusions, it seems natural to inquire in what manner the simultaneous action of the contraction of a planetary mass and of tidal friction is likely to have operated.

The subject is necessarily speculative, but the conclusions at which I arrive are, I think, worthy of notice, for although they involve much of mere conjectural assumption in respect to the quantities and amounts assumed, yet they are deduced from the rigorous dynamical principles of angular momentum and of energy.

§ 8. *On the part played by tidal friction in the evolution of planetary masses.*

To consider the subject of this section, we require--

(α) Some measure of the relative efficiency of solar tidal friction in reducing the rotational momentum and the rotation of the several planets.

(β) We have to consider the manner in which the simultaneous action of the contraction of the planetary mass and of solar tidal friction co-operate.

(γ) We have to discuss how the separation of a satellite from the contracting mass is likely to affect the course of evolution.

It is not possible to treat these questions rigorously, but without some guidance on these points further discussion would be fruitless.

The probable influence of the heterogeneity of the planetary mass on tidal friction has been already discussed, and it has been shown that the case of homogeneity will probably give good indications of the result in the true case. I therefore adhere here also to the hypothesis of homogeneity.

I will begin with (α) and consider—

The relative efficiency of solar tidal friction.

The rate at which the rotation of any one of the planets is being reduced is $\tau^2(n-\Omega)/\mathfrak{g}\mathfrak{p}$, where n, $\mathfrak{g}$, $\mathfrak{p}$ refer to the planet, and are the quantities which were previously indicated by the same symbols accented.

τ is $\frac{3}{2}M/c^3$, and therefore varies as Ω^2. With all the planets (excepting, perhaps, Mercury and Venus, according to HERSCHEL) Ω is small compared with n, and we may write n for $n-\Omega$.

It has been already shown that $\mathfrak{p}=\frac{3}{19\times 4\pi}\frac{g^2}{\mu\upsilon}$, and $\mathfrak{g}=\frac{2}{5}\frac{g}{a}$.

Hence
$$\mathfrak{p}\mathfrak{g}=\frac{2\times 3}{5\times 19\times 4\pi}\frac{g^3}{\mu a\upsilon}=\frac{2\times 3}{5\times 19\times 4\pi}\frac{\mu^2 m^3}{a^7\upsilon}$$

Therefore the rate of reduction of planetary rotation is proportional to $\Omega^4 a^7 n m^{-3}\upsilon$.

The coefficient of friction υ is quite unknown, but we shall obtain indications of the relative importance of tidal retardation in the several planets by supposing υ to be the same in all. If we multiply this expression by ma^2, we obtain an expression to which the rate of reduction of rotational momentum is proportional. By means of the data used in the preceding section I find the following results.

TABLE III.

Planet.	Number to which tidal retardation is proportional.	Number to which rate of destruction of rotational momentum is proportional.
Mercury .	1000· (?)	9·1 (?)
Venus . .	11· (?)	8·1 (?)
Earth . .	1·	1·0
Mars . .	·89	·026
Jupiter . .	·00005	2·3
Saturn . .	·000020 to ·000066	·11 to ·54

This table only refers to solar tidal friction, and the numbers are computed on the hypothesis of the identity of the coefficient of tidal friction for all the planets.

The figures attached to Mercury and Venus are open to much doubt. Perhaps the most interesting point in this table is that the rate of solar tidal retardation of Mars is nearly equal to that of the earth, notwithstanding the comparative closeness of the latter to the sun. The significance of these figures will be commented on below.

I shall now consider—

(β) *The manner in which solar tidal friction and the contraction of the planetary nebula work together.*

It will be supposed that the contraction is the more important feature, so that the acceleration of rotation due to contraction is greater than the retardation due to tidal friction.

Let h be the rotational momentum of the planet at any time; then

$$Cn=h \text{ or } n=\frac{5}{2}\frac{h}{ma^2} \quad \ldots\ldots\ldots\ldots \quad (29)$$

In accordance with the above supposition h is a quantity which diminishes slowly in consequence of tidal friction, and a diminishes in consequence of contraction, at such a rate that dn/dt is positive.

We also have
$$\mathfrak{pg}=\frac{2\times 3}{19\times 5\times 4\pi}\frac{g^3}{\mu v a}.$$

The rate of change in the dimensions of the planet's orbit about the sun remains insensible, so that τ and Ω may be treated as constant.

Then the rate of loss of rotational momentum of the planetary mass is $Cn\frac{\tau^2}{\mathfrak{gp}}\left(1-\frac{\Omega}{n}\right)$. By the above transformations we see that this expression varies as $\frac{hva}{g^3}\left(1-\frac{2}{5}\frac{\Omega m a^2}{h}\right)$.

But $m=\frac{4}{3}\pi w a^3$, and therefore $a=\left(\frac{3m}{4\pi}\right)^{\frac{1}{3}}w^{-\frac{1}{3}}$. Also $g=\mu m/a^2$.

On substituting this expression for g, and then replacing a throughout by its expression in terms of w, we see that, on omitting constant factors, the rate of loss of rotational momentum varies as $\frac{hv}{w^{\frac{7}{3}}}-\frac{kv}{w^3}$, where $k=\frac{2}{5}\left(\frac{3}{4\pi}\right)^{\frac{2}{3}}\Omega m^{\frac{5}{3}}$, a constant.

From (29) we see that if h varies as a^2, or as $w^{-\frac{2}{3}}$, n the angular velocity of rotation remains constant.

If therefore we suppose h to vary as some power of a less than 2 (which power may vary from time to time) we represent the hypothesis that the contraction causes more acceleration of rotation than tidal friction causes retardation. Let us suppose then that $h=\mathrm{H}w^{-\frac{2}{3}+\beta}$ where β is less than $\frac{2}{3}$, and varies from time to time.

Then the rate of loss of momentum varies as

$$vw^{-3}(\mathrm{H}w^{\beta}-k).$$

In order to determine the rate of loss of rotation we must divide this expression by C, which varies as $w^{-\frac{2}{3}}$.

Therefore the rate of loss of angular velocity of rotation varies as

$$vw^{-\frac{7}{3}}(\mathrm{H}w^{\beta}-k).$$

In order to determine how tidal friction and contraction co-operate it is necessary to adopt some hypothesis concerning the coefficient of friction v.

So long as the tides consist of a bodily distortion, the coefficient of friction must be some function of the density, and will certainly increase as the density increases.

Now if, as regards the loss of momentum, v varies as a power less than the cube of the density, the first factor vw^{-3} diminishes as the density increases; and if, as regards the loss of rotation, v varies as a power less than $\frac{7}{3}$ of the density, the first factor $vw^{-\frac{7}{3}}$ diminishes as the density increases.

And as the cube and $\frac{7}{3}$ powers both represent very rapid increments of the coefficient of friction with increase of density, it is probable that the first factor in both expressions diminishes as the contraction of the planetary mass proceeds.

Now consider the second factor $\mathrm{H}w^{\beta}-k$, which corresponds to the factor $n-\Omega$ in the expression for the rate of loss of rotational momentum ; Hw^8 is large compared with k so long as the rotation of the planet is fast compared with the orbital motion of the planet about the sun, and since this factor is always positive, it always increases as the contraction increases.

For planets remote from the sun, where contraction has played by far the more important part, β will be very nearly equal to $\frac{2}{3}$, and for those nearer to the sun β will be small (or it might be negative if the tidal retardation exceeds the contractional acceleration).

We thus have one factor always increasing and the other always diminishing, and the importance of the increasing factor is greater for planets remote from than for those near to the sun.

If β be small it is difficult to say how the two rates will vary as the contraction proceeds. But if β does not differ very much from $\frac{2}{3}$ both rates are probably initially small, then rise to a maximum and then diminish.

Hence it may be concluded as probable that in the history of a contracting planetary mass, which is sufficiently far from the sun to allow contraction to be a more important factor than tidal friction, both the rate of loss of rotational momentum and of loss of rotation, due to solar tidal friction, were initially small, rose to a maximum and then diminished.

These considerations are important as showing that the efficiency of solar tidal friction was probably greater in the past than at present.

We now come to (γ)—

The effect of the genesis of a satellite on the evolution.

This subject is necessarily in part obscure, and the conclusions must in so far be open to doubt.

When a satellite separates from a planetary mass, it seems probable that that part of the planetary mass, which before the change had the greatest angular momentum, is lost by the planet. Hence the rotational momentum of the planet suffers a diminution, and the mass is also diminished. An inspection of the expressions in the last paragraphs shows that it is probable that the loss of a satellite diminishes the rate of loss of planetary rotational momentum, but slightly increases the rate of loss of rotation due to solar tidal friction.

Now if the satellite be large the effect of the tides raised by the satellite in the planet is to cause a much more powerful reduction of planetary rotation than was effected by the sun. The rotational momentum thus removed from the planet reappears in the orbital momentum of its satellite. And the reduction of rotation of the planet causes a reduction of rate of solar tidal effects, by diminishing the angular velocity of the planet's rotation relatively to the sun.

The first and immediate effect of the separation of a satellite is no doubt highly speculative, but the second effect seems to follow undoubtedly, whatever be the mode of separation of the satellite.

From these considerations we may conclude that the effect of the separation of a satellite is to destroy planetary rotation, but to preserve angular momentum within the planetary subsystem.

Hence we ought to find that those planets which have large satellites have a slow rotation, but have a relatively large amount of angular momentum within their systems.

A proper method of comparison between the several planets is difficult of attainment, but these ideas seem to agree with the fact that the earth, which is large compared with Mars, rotates in the same time, but that the whole angular momentum of earth and moon is large.*

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* A method of comparing the various members of the solar system has occurred to me, but it is not founded on rigorous argument.

It seems probable that the small density of the larger planets is due to their not being so far advanced in their evolution as the smaller ones, and it is likely that they are continuing to contract and will some day be as dense as the earth.

The proposed method of comparison is to estimate how fast each of the planets must rotate if, with their actual rotational momenta, they were as condensed as the earth, and had the same law of internal density.

The period of this rotation may be called the "effective period."

With the data used above, taking the earth's mean density as unity, the mean density of Mars is ·675, that of Jupiter ·235, that of Saturn ·125 or ·111 or ·074, according to the data used.

To condense these planets we must reduce their radii in the proportion of the cube-roots of these numbers.

Their actual moments of inertia must be reduced by multiplying by the $\frac{2}{3}$rd power of these numbers, and as we suppose the law of internal density to be the same as in the earth, the moments of inertia of Jupiter and Saturn must be also increased in the proportion ·33438 to ·26138.

Then the "effective period" will be the actual period reduced by the same factors as have been given for reducing the moments of inertia.

In this way I find that the Martian day is to be divided by 1·3; the Jovian day by 2; and the Saturnian day by 3·14 to 4·44 according to the data adopted. The earth's day of course remains unchanged.

The following table gives the results.

TABLE IV.

Planet.	Actual period of rotation.	Effective period of rotation.
Earth	$23^h\ 56^m$	$23^h\ 56^m$
Mars	$24^h\ 37^m$	19^h
Jupiter	$9^h\ 55^m$	5^h
Saturn	$10^h\ 29^m$	$3^h\ 20^m$ to $2^h\ 20^m$

This seems to me to illustrate the arguments used above. For there should in general be a diminution of effective period as we recede from the sun.

It will be noted that the earth, although ten times larger than Mars, has a longer effective period. The larger masses should proceed in their evolution slower than the smaller ones, and therefore the greater proximity of the earth to the sun does not seem sufficient to account for this, more especially as it is

§ 9. *General discussion and summary.*

According to the nebular hypothesis the planets and the satellites are portions detached from contracting nebulous masses. In the following discussion I shall accept that hypothesis in its main outline, and shall examine what modifications are necessitated by the influence of tidal friction.

In § 7 it is shown that the reaction of the tides raised in the sun by the planets must have had a very small influence in changing the dimensions of the planetary orbits round the sun, compared with the influence of the tides raised in the planets by the sun.

From a consideration of numerical data with regard to the solar system and the planetary subsystems, it appears improbable that the planetary orbits have been much enlarged by tidal friction, since the origin of the several planets. But it is possible that part of the eccentricities of the planetary orbits is due to this cause.

We must therefore examine the several planetary subsystems for the effects of tidal friction.

From arguments similar to those advanced with regard to the solar system as a whole, it appears unlikely that the satellites of Mars, Jupiter, and Saturn originated very much nearer the present surfaces of the planets than we now observe them. But the data being insufficient, we cannot feel sure that the alteration in the dimensions of the orbits of these satellites has not been considerable. It remains, however, nearly certain that they cannot have first originated almost in contact with the present surfaces of the planets, in the same way as, in previous papers, has been shown to be probable with regard to the moon and earth.

The numerical data in Table II., § 7, exhibit so striking a difference between the terrestrial system and those of the other planets, that, even apart from the considerations adduced in this and previous papers, we should have grounds for believing that the modes of evolution have been considerably different.

This series of investigations shows that the difference lies in the genesis of the moon close to the present surface of the planet, and we shall see below that solar tidal

shown above that the efficiency of solar tidal friction is of about the same magnitude for the two planets. It is explicable however by the considerations in the text, for it was there shown that a large satellite was destructive of planetary rotation.

If we estimate how fast the earth must rotate in order that the whole internal momentum of moon and earth should exist in the form of rotational momentum, then we find an effective period for the earth of $4^h\ 12^m$. This again illustrates what was stated above, viz.: that a large satellite is preservative of the internal momentum of the planet's system.

The orbital momentum of the satellites of the other planets is so small, that an effective period for the other planets, analogous to the $4^h\ 12^m$ of the earth, would scarcely differ sensibly from the periods given in the table.

If Jupiter and Saturn will ultimately be as condensed as the earth, then it must be admitted as possible or even probable that Saturn (and perhaps Jupiter) will at some future time shed another satellite; for the efficiency of solar tidal friction at the distance of Saturn is small, and a period of two or three hours gives a very rapid rotation.

friction may be assigned as a reason to explain how it happened that the terrestrial planet had contracted to nearly its present dimensions before the genesis of a satellite, but that this was not the case with the exterior planets.

The numbers given in Table III., § 8, show that the efficiency of solar tidal friction is very much greater in its action on the nearer planets than on the further ones. But the total amount of rotation of the various planetary masses destroyed from the beginning cannot be at all nearly proportional to the numbers given in that table, for the more remote planets must be much older than the nearer ones, and the time occupied by the contraction of the solar nebula from the dimensions of the orbit of Saturn down to those of the orbit of Mercury must be very long. Hence the time during which solar tidal friction has been operating on the external planets must be very much longer than the period of its efficiency for the interior ones, and a series of numbers proportional to the total amount of rotation destroyed in the several planets would present a far less rapid decrease, as we recede from the sun, than do the numbers given in Table III. Nevertheless the disproportion between these numbers is so great that it must be admitted that the effect produced by solar tidal friction on Jupiter and Saturn has not been nearly so great as on the interior planets.

In § 8 it has been shown to be probable that, as a planetary mass contracts, the rate of tidal retardation of rotation, and of destruction of rotational momentum increases, rises to a maximum, and then diminishes. This at least is so, when the acceleration of rotation due to contraction exceeds the retardation due to tidal friction; and this must in general have been the case. Thus we may suppose that the rate at which solar tidal friction has retarded the planetary rotations in past ages was greater than the present rate of retardation, and indeed there seems no reason why many times the present rotational momenta of the planets should not have been destroyed by solar tidal friction. But it remains very improbable that so large an amount of momentum should have been destroyed as to materially affect the orbits of the planets round the sun.

I will now proceed to examine how the differences of distance from the sun would be likely to affect the histories of the several planetary masses.

According to the nebular hypothesis a planetary nebula contracts, and rotates quicker as it contracts. The rapidity of the revolution causes its form to become unstable, or, perhaps a portion gradually detaches itself; it is immaterial which of these two really takes place. In either case the separation of that part of the mass, which before the change had the greatest angular momentum, permits the central portion to resume a planetary shape. The contraction and increase of rotation proceed continually until another portion is detached, and so on. There thus recur at intervals a series of epochs of instability or of abnormal change.

Now tidal friction must diminish the rate of increase of rotation due to contraction, and therefore if tidal friction and contraction are at work together, the epochs of instability must recur more rarely than if contraction acted alone.

If the tidal retardation is sufficiently great, the increase of rotation due to contraction will be so far counteracted as never to permit an epoch of instability to occur.

Now the rate of solar tidal frictional retardation decreases rapidly as we recede from the sun, and therefore these considerations accord with what we observe in the solar system.

For Mercury and Venus have no satellites, and there is a progressive increase in the number of satellites as we recede from the sun. Moreover, the number of satellites is not directly connected with the mass of the planet, for Venus has nearly the same mass as the earth and has no satellite, and the earth has relatively by far the largest satellite of the whole system. Whether this be the true cause of the observed distribution of satellites amongst the planets or not, it is remarkable that the same cause also affords an explanation, as I shall now show, of that difference between the earth with the moon, and the other planets with their satellites, which has caused tidal friction to be the principal agent of change with the former but not with the latter.

In the case of the contracting terrestrial mass we may suppose that there was for a long time nearly a balance between the retardation due to solar tidal friction and the acceleration due to contraction, and that it was not until the planetary mass had contracted to nearly its present dimensions that an epoch of instability could occur.

It may also be noted that if there be two equal planetary masses which generate satellites, but under very different conditions as to the degree of condensation of the masses, then the two satellites so generated would be likely to differ in mass; we cannot of course tell which of the two planets would generate the larger satellite. Thus if the genesis of the moon was deferred until a late epoch in the history of the terrestrial mass, the mass of the moon relatively to the earth, would be likely to differ from the mass of other satellites relatively to their planets.

If the contraction of the planetary mass be almost completed before the genesis of the satellite, tidal friction, due jointly to the satellite and to the sun, will thereafter be the great cause of change in the system, and thus the hypothesis that it is the sole cause of change will give an approximately accurate explanation of the motion of the planet and satellite at any subsequent time.

That this condition is fulfilled in the case of the earth and moon, I have endeavoured to show in the previous papers of this series.

At the end of the last of those papers the systems of the several planets were reviewed from the point of view of the present theory. It will be well to recapitulate shortly what was there stated and to add a few remarks on the modifications and additions introduced by the present investigation.

The previous papers were principally directed to the case of the earth and moon, and it was there found that the primitive condition of those bodies was as follows:—the earth was rotating, with a period of from two to four hours, about an axis inclined at 11° or 12° to the normal to the ecliptic, and the moon was revolving, nearly in contact with the earth, in a circular orbit coincident with the earth's equator, and with a periodic time only slightly exceeding that of the earth's rotation.

Then it was proved that lunar and solar tidal friction would reduce the system from this primitive condition down to the state which now exists by causing a retardation of terrestrial rotation, an increase of lunar period, an increase of obliquity of ecliptic, an increase of eccentricity of lunar orbit, and a modification in the plane of the lunar orbit too complex to admit of being stated shortly.

It was also found that the friction of the tides raised by the earth in the moon would explain the present motion of the moon about her axis, both as regards the identity of the axial and orbital revolutions, and as regards the direction of her polar axis.

Thus the theory that tidal friction has been the ruling power in the evolution of the earth and moon completely coordinates the present motions of the two bodies, and leads us back to an initial state when the moon first had a separate existence as a satellite.

This initial configuration of the two bodies is such that we are almost compelled to believe that the moon is a portion of the primitive earth detached by rapid rotation or other causes.

There may be some reason to suppose that the earliest form in which the moon had a separate existence was in the shape of a ring, but this annular condition precedes the condition to which the dynamical investigation leads back.

The present investigation shows, in confirmation of preceding ones,[*] that at this origin of the moon the earth had a period of revolution about the sun shorter than at present by perhaps only a minute or two, and it also shows that since the terrestrial planet itself first had a separate existence the length of the year can have increased but very little—almost certainly by not so much as an hour, and probably by not more than five minutes.[†]

With regard to the 11° or 12° of obliquity which still remains when the moon and earth are in their primitive condition, it may undoubtedly be partly explained by the friction of the solar tides before the origin of the moon, and perhaps partly also by the simultaneous action of the ordinary precession and the contraction and change of ellipticity of the nebulous mass.[‡]

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[*] "Precession," § 19.

[†] If the change has been as much as an hour the rotational momentum of the earth destroyed by solar tidal friction must have been 33 times the present total internal momentum of moon and earth. For the orbital momentum of a planet varies as the cube root of its periodic time, and if we differentiate logarithmically we obtain the increment of periodic time in terms of the increment of orbital momentum. Then taking the numerical data from Tables I. and II. we see that this statement is proved by the fact that 3×33 times $[216 \div \cdot 01720 \times 10^{10}] \times 365 \cdot 25 \times 24$ is very nearly equal to unity.

[‡] See a paper "On a Suggested Explanation of the Obliquity of Planets to their Orbits," 'Phil. Mag.,' March, 1877. See however § 21 "Precession."

In the review referred to I examined the eccentricities and inclinations of the orbits of the several other satellites, and found them to present indications favourable to the theory. In the present paper I have given reasons for supposing that the tidal friction arising from the action of the other satellites on their planets cannot have had so much effect as in the case of the earth. That those indications were not more marked, and yet seemed to exist, agrees well with this last conclusion.

The various obliquities of the planets' equators to their orbits were also considered, and I was led to conclude that the axes of the planets from Jupiter inwards were primitively much more nearly perpendicular to their orbits than at present. But the case of Saturn and still more that of Uranus (as inferred from its satellites) seem to indicate that there was a primitive obliquity at the time of the genesis of the planets, arising from causes other than those here considered.

The satellites of the larger planets revolve with short periodic times; this admits of a simple explanation, for the smallness of the masses of these satellites would have prevented tidal friction from being a very efficient cause of change in the dimensions of their orbits, and the largeness of the planets' masses would have caused them to proceed slowly in their evolution.

If the planets be formed from chains of meteorites or of nebulous matter the rotation of the planets has arisen from the excess of orbital momentum of the exterior over that of the interior matter. As we have no means of knowing how broad the chain may have been in any case, nor how much it may have closed in on the sun in course of concentration, we have no means of computing the primitive angular momentum of a planet. A rigorous method of comparison of the primitive rotations of the several planets is thus wanting.

If however the planets were formed under similar conditions, then, according to the present theory, we should expect to find the exterior planets now rotating more rapidly than the interior ones. It has been shown above (see Table IV., note to § 8) that, on making allowance for the different degrees of concentration of the planets, this is the case.

That the interior satellite of Mars revolves with a period of less than a third of its planet's rotation is perhaps the most remarkable fact in the solar system. The theory of tidal friction explains this perfectly,* and we find that this will be the ultimate fate of all satellites, because the solar tidal friction retards the planetary rotation without directly affecting the satellite's orbital motion.

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* It is proper to remark that the rapid revolution of this satellite might perhaps be referred to another cause, although the explanation appears very inadequate.

It has been pointed out above that the formation of a satellite out of a chain or ring of matter must be accompanied by a diminution of periodic time and of distance. Thus a satellite might after formation have a shorter periodic time than its planet.

If this, however, were the explanation, we should expect to find other instances elsewhere, but the case of the Martian satellite stands quite alone.

[I believe that I now (July, 1881) see some reason to suppose that the earliest form of a satellite may not be annular. The investigation necessary to test this idea seems likely to prove a difficult one.]

The numerical comparison in Table III. shows that the efficiency of solar tidal friction in retarding the terrestrial and Martian rotations is of about the same degree of importance, notwithstanding the much greater distance of the planet Mars.

From the discussion in this paper it will have been apparent that the earth and moon do actually differ from the other planets in such a way as to permit tidal friction to have been the most important factor in their history.

By an examination of the probable effects of solar tidal friction on a contracting planetary mass, we have been led to assign a cause for the observed distribution of satellites in the solar system, and this again has itself afforded an explanation of how it happened that the moon so originated that the tidal friction of the lunar tides in the earth should have been able to exercise so large an influence.

In this summary I have endeavoured not only to set forth the influence which tidal friction may, and probably has had in the history of the system, but also to point out what effects it cannot have produced.

The present investigations afford no grounds for the rejection of the nebular hypothesis, but while they present evidence in favour of the main outlines of that theory, they introduce modifications of considerable importance.

Tidal friction is a cause of change of which LAPLACE's theory took no account,* and although the activity of that cause is to be regarded as mainly belonging to a later period than the events described in the nebular hypothesis, yet its influence has been of great, and in one instance of even paramount importance in determining the present condition of the planets and their satellites.

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* Note added on July 28, 1881.

Dr. T. R. MAYER appears to have been amongst the first, if not quite the first, to draw attention to the effects of tidal friction. I have recently had my attention called to his paper on "Celestial Dynamics" [Translation, 'Phil. Mag.,' 1863, vol. 25, pp. 241, 387, 417], in which he has preceded me in some of the remarks made above. He points out that, as the joint result of contraction and tidal friction, "the whole life of the earth therefore may be divided into three periods—youth with increasing, middle age with uniform, and old age with decreasing velocity of rotation."

X *On the Thermal Conductivity of Water.*

By J. T. Bottomley.

Communicated by Sir William Thomson, *LL.D., F.R.S., Professor of Natural Philosophy in the University of Glasgow.*

Received March 11,—Read April 3, 1879.

The experiments on the thermal conductivity of water, of which an account is given in the following paper, were undertaken at the wish of Sir William Thomson, and by a method devised by him some years ago. They have been carried out in several successive Winter Sessions in the Physical Laboratory of the University of Glasgow, with the assistance of students and experimental scholars, among whom I must mention specially Mr. J. Reid, Jun., and Mr. M. T. Brown.

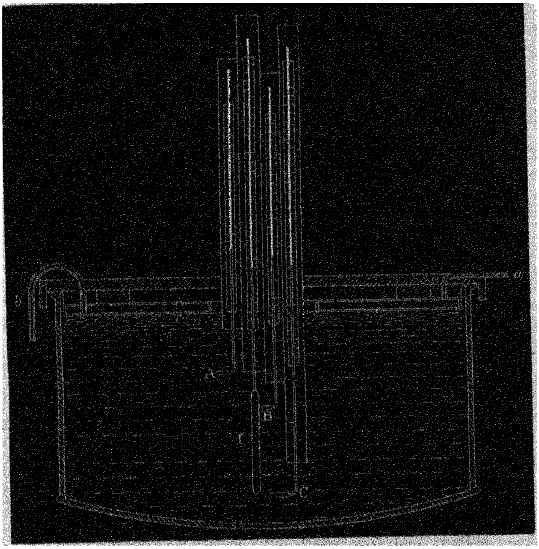
Description of apparatus.—Figs. 1 and 2 show the arrangement of apparatus for experiments on the thermal conductivity of a liquid. In each case I have a vessel or tank so large that the heat lost at the sides does not affect sensibly the condition of the central portion of the liquid, either directly, or by convection currents set up close to the sides. The water or other liquid to be experimented on is contained in this vessel and is heated from above.

The tank, fig. 1, is a cylindrical vessel of thin sheet iron. It is 120 centims. in diameter and 60 centims. high. At the top there is a hollow cover in contact with the water in the tank, and a current of steam is kept blowing through this hollow cover. The steam enters by the pipe *a* and is blown out by the pipe *b*, which is made to dip down to the bottom of the steam chamber in such a way that the water formed in the chamber by the condensation of the steam is blown out through this pipe along with the surplus steam. In the middle of the hollow cover, or steam chamber, there is a rectangular opening, walled in so as to make the chamber steam-tight, and the stems of the thermometers employed in the experiments pass up through this opening. The water being heated from above transmits the heat downward to the layers beneath.

The tank shown in fig. 2 is a wooden vessel, an ordinary wine cask, about 64 centims. in diameter, and 90 centims. high. It had the disadvantage of not being uniform in section from top to bottom, but this had been done away with in tanks recently constructed. I require a *wooden* vessel for experimenting on solutions of salts in water. Solution of sulphate of zinc, for example, quickly corrodes and eats its way through an iron vessel; and solution of sulphate of copper (at any rate solution of the sulphate of copper of commerce) eats through both iron and copper.

The mode of applying heat in this second apparatus is different from that described above. When an experiment is to be made the tank is filled with water (or other liquid) up to the level, D, and after the water has been carefully stirred from top to bottom and has come to rest after the stirring, a "floater" is placed on the surface. The floater is a large thin board with its edges carefully bevelled all round. It covers nearly the whole surface of the liquid below. Hot water is allowed to flow, slowly at first, but afterwards as quickly as is convenient, on the top of the floater, and a thick layer of very hot water is laid on the top of the colder water beneath. The edges of the floater are bevelled so that the hot water falling on it is shot off horizontally at the edges without any downward motion, and therefore with no tendency to mingle with the colder water, of greater density, below; and if the operation of laying on a layer of hot water is carefully carried out, there is scarcely any mixing of the hot water with the cold. This can be seen if the experiment is tried in a large glass vessel. A layer of hot water can easily be deposited on the top of a quantity of cold water, with so little mixing that the layer of separation intermediate between the hot and the cold, which are distinguishable by the difference in refractive indices of hot and cold water, is *seen* to be not thicker than a sheet of paper. I have also tried experiments as to mixing on a large scale in the sheet-iron tank, fig. 1. On running

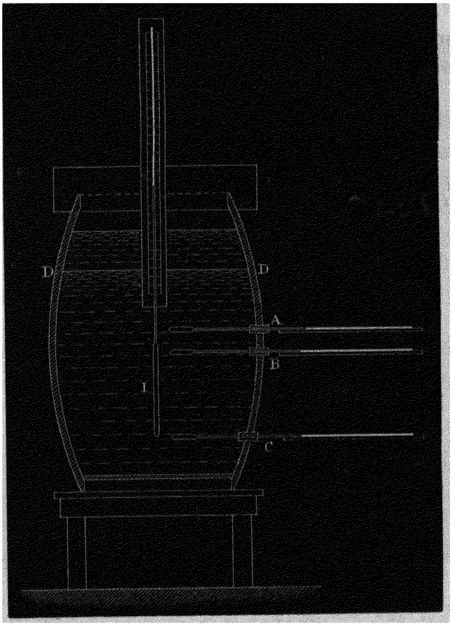
Fig. 1.



a layer of hot water on the top of a quantity of cold with the aid of a properly constructed floater, and applying the finger outside at the place of separation, a space of separation of the hot from the colder water can, even in this rough way, be ascertained to be certainly not so thick as half a centimetre.

Figs. 1 and 2 show also the arrangement of the thermometers employed. In each case four are used. Of these three have their bulbs horizontal. Each of these shows the temperature of the layer in which it is placed. The thermometers A and B in the diagrams show the difference of temperature between the two sides of a stratum of the liquid a few centimetres (5 to 10 centims.) thick. The thermometer C is merely used to show that heat has been conducted down to the bottom of the portion of liquid under experiment, and is therefore being conducted out below. When the thermometer C begins to rise the experiment is at an end.

Fig. 2.



The fourth thermometer, marked I in the diagrams, is called the *integrating thermometer*. It has a bulb 30 centims. long, which is placed vertically in the liquid. This thermometer is intended to show the whole quantity of heat that passes during any time into the stratum of the liquid bounded by horizontal planes through the top and bottom of its bulb.

When heat is being conducted downward through the liquid, the various parts of the integrating thermometer are at different temperatures, the temperatures decreasing from above downwards. On the whole, however, the temperature indicated by the integrating thermometer is the *average* temperature from top to bottom of the layer of liquid which contains the bulb of the thermometer.

I have verified by experiments that the integrating thermometer does give with very considerable accuracy the average temperature from top to bottom of a cylindrical layer of liquid in which it is contained. For this purpose I placed the thermometer vertical in a cylindrical vessel, the bottom of the bulb being at the bottom of the vessel. The bottom of the vessel was plane and was horizontal. I then poured in an inch or two of cold water. On the top of that layer I deposited a layer of warmer water, then a layer of still warmer water, and so on, making a succession of layers of water of gradually increasing temperature till the level of the top of the bulb was reached. I then read off the temperature indicated by the thermometer as the average of the whole. I next stirred the whole from top to bottom so as to equalize the temperature throughout and read the thermometer again. Many such experiments were tried, and in every case I found that the alteration in the reading was very small, and was certainly not more than variation which could be accounted for by unavoidable errors introduced by changes of temperature of the parts of the containing vessel and of the stirrer.* These tests were so severe in comparison with the requirements of the present experiments on the conductivity of heat by liquids, that I am satisfied that no error of consequence, in comparison with uncertainties that I have not yet been able to get rid of, is introduced by want of *perfect* averaging by the integrating thermometer.

From the description of the arrangements for these experiments, it will be seen that we have a layer of liquid, of known thickness, between the thermometers A and B kept with one of its sides hotter than the other. The thermometers A and B give the temperatures of the two sides of the layer. The heat conducted across it warms the liquid beneath, and the whole heat that flows across the layer between A and B in any given time is known from the readings of the integrating thermometer at the beginning and end of the interval. Now if the temperature of the layers at A and B could be kept constant during an experiment the calculation of the thermal conductivity of the liquid from these data would be of the simplest possible kind.

The quantity of heat that passes across the stratum between A and B in any time is in simple proportion to the difference of temperatures between the two sides of the stratum; it is in simple proportion to the time; it is in simple proportion to the area of the stratum; it is inversely proportional to the cross-sectional *area* of the stratum;

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* It is hardly necessary to say that alteration of specific heat of the water by change of temperature could not be taken into account in considering the results of experiments such as those described.

and it is in simple proportion to the thermal conductivity of the liquid. Now let T and T′ denote the temperatures of A and B respectively—supposed for the moment constant—let a denote the thickness of the layer between A and B, let t denote the time in seconds, k the conductivity, and Q the quantity of heat conducted across the stratum whose area is A;—then we should have

$$Q=k\frac{T-T'}{a}At \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, taking as unit quantity of heat the quantity required to raise unit bulk of water one degree, and neglecting for the present differences of specific heat, in the different layers of the liquid column, due to differences of temperature, we have, if l be the length of the integrating thermometer, D the difference between the average temperature from top to bottom at the beginning and the average temperature from top to bottom at the end, and if c be the specific heat of the liquid per unit of bulk

$$Q=A.l.c.D \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, equating (1) and (2), we have

$$k\frac{T-T'}{a}.A.t=A.l.c.D$$

or

$$\frac{k}{c}=\frac{Dl}{\frac{T-T'}{a}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$k=\frac{Dlc}{\frac{T-T'}{a}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In calculating from the results of experiments on the conductivity of water I have taken its specific heat as being equal to 1. The experiments that I have made up to the present time do not permit of my being able to take into account either variations of specific heat in the different parts of the layer tested, or variations of the average specific heat in the layer from top to bottom from that of water at the temperature chosen for defining unit specific heat. The formula, therefore, which would be used for calculation, if the conditions mentioned above—namely, the constancy during an interval of time of the temperature at the levels A and B—were fulfilled, is

$$k=\frac{Dl}{\frac{T-T'}{a}t}.$$

It will be seen, however, from the method of making the experiments that to fulfil this condition was impossible.

In both of the modes of conducting the experiments the heating arrangement is at a considerable distance from the layers which the thermometers A and B represent; and the heat supplied travels gradually downward.

In the arrangement (fig. 1) the rise of the two thermometers A and B goes on continuously. In the arrangement of fig. 2, where there was only a limited supply of heat, namely, that belonging to the layer of hot water poured on the top, both thermometers first gradually rise in temperature and then gradually fall.

The following, then, was the mode of experimenting and of calculation. The water to be experimented on was first carefully stirred from top to bottom to equalize the temperature throughout. The heating was applied and the temperatures of all four thermometers were read. An hour or so afterwards the regular observations began. After that time the thermometer A was generally found to be commencing to rise slowly. Somewhat later the thermometers I and B were found to be rising, and from that time until the lowest thermometer was observed to be rising the reading of the thermometers was carried on continuously at intervals of five minutes or of ten minutes. The observations are generally carried on for five or six hours and are therefore laborious.

It is unnecessary to put down here the readings taken in the various experiments carried out. It is only a small portion of the series of numbers recorded in any one experiment that is made use of for calculation. The part that is made use of is that which keeps the difference of temperatures of the thermometers A and B as nearly as possible constant for a considerable time. In the first mentioned method of heating it is that part at which the two thermometers A and B are both rising at nearly the same rate. In the second method it is that part of the time of observation during which the two thermometers remain most nearly stationary. During this part of the time the increase in temperature of the integrating thermometer is taken. The *average* temperature indicated by A is taken as the temperature of the level which is represented by it, and the *average* temperature of the thermometer B is taken as the temperature of the level which is represented by it. These are used in the formula given above as the temperatures T and T′ respectively.

A specimen of part of a series of thermometer readings, extending over two hours, is given in Table I. appended. The heating was carried on by the method of fig. 1. The heating was commenced at about 9 A.M., and the observations were carried on till 4 P.M., when they were discontinued as the thermometer at the bottom of the stratum occupied by the integrator began to rise very sensibly.

Table II. contains results calculated from a number of satisfactory experiments. From these and from the consideration of other results, I conclude that the conductivity for heat of water at the temperature at which I have experimented is to be reckoned at ·0023 square centims. per second.

The temperatures at which the experiments were made have varied from 12° C. to 17° C. I have not yet made any attempt to trace difference in conductivity corresponding with different temperatures of the water.

I have also commenced experimenting on the conductivity for heat of saline solutions, but I am not yet prepared to give numerical results that I should consider satisfactory. In order to make a comparison between the conductivity of water, and that of a liquid other than water, it is necessary to know the specific heat of the liquid, or, in the case of a saline solution, of the saline solution at the density which it has in the experiment on conductivity. Solution of sulphate of zinc has been experimented on in a preliminary way for conductivity, and its specific heat at various densities has been determined. I hope, however, with improved apparatus furnished by means of a grant from the Government Fund of £4,000, to be able soon to give much better results than I can offer at present.

Table I.—Conduction of heat by water. Series of thermometer readings taken with arrangement of fig. 1, December 10, 1873. Time, 1.30 p.m., to 3.30 p.m. Thermometers A, B, I, C, as shown in fig. 1.

Time.	Thermometers.			
	A.	B.	I.	C.
1·30	140·5	113·05	38·95	61·7
·35	143·1	113·8	39·35	
·40	145·0	114·9	40·15	
·45	147·2	115·75	40·8	
·50	148·45	116·5	41·05	
·55	149·45	116·5	41·15	
2·0	151·6	117·15	41·9	
·5	152·95	118·15	42·3	
·10	154·45	118·85	42·65	
·15	157·95	119·85	43·05	
·20	159·95	120·5	43·75	
·25	161·95	121·5	44·05	
·30	163·4	122·0	44·6	61·75
·35	166·3	122·6	45·1	
·40	168·0	123·6	45·9	
·45	169·3	124·3	46·25	
·50	170·0	124·65	46·6	
·55	170·1	124·85	46·65	
3·0	170·95	125·45	47·05	
·5	174·0	126·85	47·95	
·10	176·05	128·0	48·5	
·15	178·2	129·0	49·15	
·20	180·3	129·65	49·95	
·25	181·5	130·2	50·05	
·30	182·6	130·85	50·6	62·05

TABLE II.—Results of experiments on thermal conductivity of water, expressed in square centimetres per second.

	Date and hour of experiment.	Results.	Remarks. Heating commenced always about 9 A.M.
1	1873, Dec. 3, 1 P.M. to 2 P.M. . . .	·00209	Results 1–13 obtained by method of fig. 1.
2	" " 9, 10.15 A.M. to 11.15 A.M.	·00276	Probably too high a result owing to some disturbance at the beginning of the experiment.
3	" " " 11.20 A.M. to 12.20 P.M.	·00226	
4	" " " 12.25 P.M. to 1.25 P.M. .	·00212	
5	" " " 1.30 P.M. to 2.30 P.M. .	·00230	
6	. . . .	·00235	Other results calculated from observations of December 9th taking different intervals of time during the experiment.
7	. . . .	·00230	
8	. . . .	·00236	
9	. . . .	·00228	
10	1873, Dec. 10, 11 A.M. to 12 NOON . .	·00244	
11	" " " 12 NOON to 1 P.M. . .	·00218	
12	" " " 1.30 P.M. to 2.30 P.M. .	·00229	
13	" " " 2.30 P.M. to 3.30 P.M. .	·00231	
14	1876, March 24, 2 P.M. to 3 P.M. . .	·00197	Results 14–16 obtained by method of fig. 2. The results given by this method have uniformly given a smaller number for the conductivity than that given by the first method. I attribute this to one or both of two causes. First, the supply of heat is limited and is not large enough in the arrangements that I had at my command when these experiments were made, and, secondly, the vessel in which the experiments were made was not, I believe, large enough to make sure of no loss by the sides. Some experiments that I made in a much smaller vessel some time ago pointed clearly to this cause, giving me a very low number for the value of k. I hope with improved apparatus to make sure of the result; but in the meantime I believe it to be about 0·002 of a square centimetre per second.
15	" " " 3 P.M. to 4 P.M. . .	·00212	
16	" " 31, 1.40 P.M. to 2.40 P.M.	·00194	

ADDITIONAL NOTE.

(Added December, 1880.)

In the foregoing paper I have supposed that by taking a vessel of very large horizontal cross section I should be able to eliminate the sensible effect of conduction of heat downwards by the sides of the vessel. However, considering the importance of the part which the sides might play, and the statements in books and published writings that the conduction of heat downwards in liquids, observed by DEPRETZ and others, was wholly, or nearly wholly, due to conduction by the sides of the containing vessel, it was considered advisable to put the matter to direct test. The following experiments were accordingly made at the suggestion of Professor STOKES in the months of March and April last.

In order to test any possible effect producible by the conduction of heat downwards by the sides of the vessel and by the convection currents thereby established, the following arrangement was made. Within the cylindric tank in which the testing was carried on, a circular screen of thin cotton material was hung, which was everywhere 6½ centims. from the outer walls, and which extended from top to bottom of the tank. Any convective action due to conduction of heat downwards by the walls of the vessel would go on outside the screen: and the enclosed column, 77 centims. in diameter, would be but little affected by these slight motions. The thermometers projected into the middle of the enclosed column through small holes cut in the screen.

Experiments made in March and April, 1880, with and without the screen, gave the following numbers for the apparent thermal conductivity of the water (in terms of an undetermined unit):—

without screen
·00137, ·00135, ·00130;
with screen
·00154, ·00149, ·00158.

These numbers, though not in complete agreement, are yet quite of the same order. They seem to indicate that the screen does produce some effect, and I propose to carry out a much more complete series of experiments in order to determine exactly to what this is due. One of the greatest difficulties of the investigation is the determination of what is to be considered as the effective distance between the levels represented by the thermometers A and B in the diagram given in the foregoing paper; and it prevented me from reducing the results of these last experiments satisfactorily to absolute measure, though it does not at all affect their *relative* values. Any error in estimating this distance produces a proportionate error in the absolute value of the numerical result. I am therefore making preparations to eliminate as far as I can this source of error. I believe however that the numbers given in Table II. may be taken as approximations to the absolute conductivity of water, which has not been hitherto even roughly determined.

XI. *Description of some Remains of the Gigantic Land-Lizard* (Megalania prisca, Owen), *from Australia.*—Part III.

By Professor Owen, *C.B., F.R.S., &c.*

Received January 20,—Read February 3, 1881.

[Plates 64–66.]

Mr. George Frederick Bennett, pursuing, after transmission of the subjects of Part II.,* his exploration of the drift-bed of "King's Creek," succeeded in extricating other fossils which, from their size and contiguity with those previously exhumed, he concluded to be portions of the same individual.

On the reception and readjustment of these supplementary fossils, which, like their predecessors, reached me in a fragmentary condition, my surprise at the additional armature of the animal, so exemplified, exceeded that with which I contemplated the evidences of the cranial weapons of the great extinct Lacertian.

If, indeed, the last received fossils had first come to hand a conclusion that they formed part of some huge Armadillo might have been condoned.

They consist of portions of the armour of a tail, and include three annular segments and the terminal cap of an osseous sheath, of which the two hindmost rings had coalesced with each other and the cap (Plate 65, figs. 1, 2, and 3). The anterior detached ring (ib., *a*, and fig. 4) may have come from a more advanced part of the tail, but the peripheral border of the hinder aperture (ib., fig. 4, *e*, *e*) fits that of the front aperture of the foremost of the coalesced group (ib., figs. 1–3, *a'*, *b*).

Each of the annular segments sends off two pairs of massive conical processes, like the horn-cores of the skull, but of larger size, being broader and thicker in proportion to their length and rather more obtuse at the apex. The cone forming the tip of the tail-sheath (ib., figs. 1–3, *c*) is simple and cap-shaped.

The area of the detached ring (Plate 64) is of a wide, vertically oblong, or ovoid shape, 6 inches in vertical, 5 inches in transverse, diameters, with a moderately smooth inner surface. A feebly developed medial longitudinal ridge projects downward or centrad from the upper part or key-stone of the arch. The outside transverse diameter of the segment, excluding the well-defined processes, is 6½ inches, the antero-posterior diameter or length of the ring is 6½ inches. The thickness of the bony wall near the hind border is from 8 to 10 lines, decreasing a little toward that margin;

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* Phil. Trans., Vol. 171, 1880, p. 1037.

the front border thins off almost to an edge. This border was received by the hind one of the segment in advance; the hind border, though somewhat mutilated, fits very fairly the front border of the foremost of the coalesced segments; in its present state it is, in great part, formed by the hind borders of the bases of the two pairs of solid conical processes.

Of these the upper or dorsal pair (Plate 65, figs. 1–4, *a, a*) are the largest and longest; the transverse diameter of the base of each is 4 inches; the antero-posterior diameter is 3¾ inches; the height, taken from the outer border of the base of the cone, is 4 inches. The body of the cone is subtrihedral with the apex directed obliquely upward, outward, and backward. The medial and hinder facets are convex both across and lengthwise; the antero-external facet is partly flattened across and is concave lengthwise. The surface of this horn-core is roughened by numerous small neuro-vascular foramina, like those upon the cephalic weapons, indicative of the activity of the sheath-forming periosteum. The second pair, or lateral cores (ib., ib., *b, b*) are much smaller, and are mammilloid in shape. Each measures 2½ inches across the base and rises to the same height, measured along the outer basal border. The hind border of the ring, as here conserved, is continued directly upon the base of the core; the extension thereto from the fore border is more gradual. From tip to tip of the dorsal cores (*a, a*) is 10 inches; the same admeasurement of the lateral pair (*b, b*) gives 9 inches.

The cavity of the readjusted portions of the free caudal dermosteal girdle contained no corresponding endo-skeletal segment, nor any adherent parts of such. The vertebra had probably escaped the research of my friend.

The anterior aperture of the antepenultimate coalesced segment exposed the corresponding caudal vertebra (Plate 64).

The key-stone ridge, answering to that noted in the foregoing segment, was here confluent with the neural spine (*ns.*) of its vertebra; a like confluence attached the hæmal spine (*hs.*) to the floor. Thus the area of the girdle was bisected by the vertebra. But the presence of a ridge (*e, e*) from each side of the inner surface of the girdle, and the fractured ends of answerable transverse processes (*d, d*) directed thereto, indicated that the surrounding wall had been strengthened by both transverse and vertical buttresses. The centrum of this caudal vertebra is compressed and is directly continued, by coalescence, with the bases of both neur- and hæm-apophyses. The centrum contracts to both these junctions, and narrows transversely to its exposed anterior part, the broken surface of which indicates bony union with a caudal vertebra in advance. The neural canal (*m*) has a transverse diameter of 5 lines, the vertical diameter is less: a low ridge projects from the floor of the canal. The section of the hæmal canal (*p*) gives a vertical ellipse 3 lines by 1½ lines.

The inner surface of the bony girdle is moderately smooth, as in the anterior segment: the outer surface differs chiefly in the larger size of the lateral horn-cores (*b′*). The upper or dorsal pair (Plate 64, *a′*) present the same size as in the free segment, with the same general form and direction, but sloping a little more outwards, so that their inner or medial surfaces meet along the upper mid-line of the supporting ring at a more open angle (Plate 64, *a′*, *a′*). The antero-posterior diameter of the base of each core is 4½ inches, and is coextensive with that dimension of the supporting girdle: the transverse basal diameter of each upper core is the same: the height or length of the core taken along the upper medial line is 5½ inches; taken along the outer side of the cone it is 4½ inches.

Of the lateral horn-cores (*b′*, *b′*) the fore-and-aft length of the base is co-extensive with that of its supporting ring: the height of the core from the upper border of the base is 2 inches. The apex is less obtuse than that of the antecedent lateral core. The under side or basal border of the core *b′* is directly continued into the under and outer surface of the girdle. The outside vertical diameter of the antepenultimate girdle is 6½ inches, its transverse diameter taken at the interspace of the dorsal and lateral cores is 7 inches. The same diameter of the area of the girdle is 5 inches; its vertical diameter is 4½ inches. The contour of the anterior outlet is subquadrate with the angles broadly rounded off, and the sides bulging a little inward opposite the caudal diapophyses.

The penultimate segment presents a similar structure to the preceding, with some loss of depth and breadth, and a little increase of length, viewed laterally, as in Plate 65, fig. 1. The homologous processes, or horn-cores, are indicated by the letters *a″* and *b″* in figures 1–3 of that Plate. The apex of *a″* is directed backward and outward, that of *b″* is relatively less produced than in the antecedent segment.

The terminal segment (ib. ib., *c*) is a simple cone, with a base of 3 inches in both transverse and vertical diameters, and a subobtuse apex directed backward.

At the under surface of the preserved parts of the tail-sheath (ib., fig. 3) the last two segments have the line of confluence more feebly marked than above or at the sides, their respective contributions to the under surface being indicated by a shallow linear transverse depression. The under surface of the antepenultimate segment (ib., *b**) is broadly heptagonal, with the anterior transverse border forming a low angle forward. The corresponding part of the free segment is, in great part, broken away. The whole of the preserved under surface of the above portion of the tail-sheath is flattened and shows marks of attrition against the surface on which the *Megalania* trod, as if ossification had there extended into the dermal coat.

That the horny sheaths of the above-described bony supports or cores arming the end of the tail may have been applied to deliver blows upon an assailant, seems not improbable, and this part of the organisation of the great extinct Australian Dragon may be regarded, with the cranial horns, as parts of both an offensive and defensive apparatus.

As with the cranial weapons, so with the caudal ones, examination and comparison were, in the first place, instituted in relation to all available specimens of existing *Lacertilia;* and the most notable instance of tail-armature seemed to be that of a small

Uromastix, from Zanzibar, recently described and figured as *U. princeps* by my friend and colleague A. W. E. O'SHAUGHNESSY, Esq., in the 'Proceedings of the Zoological Society of London,' for June 1, 1880 (p. 445, plate 43). The striking character and proportions of the caudal weapons of this Lizard may, perhaps, weigh for admission of a copy of the above-cited figure, amongst the illustrations of the present paper (Plate 66, fig. 1). In this recent instance, however, the horns, or horny spines, are confined to the tail.

As in *Megalania* they have an annular arrangement and are best developed on the dorsal and dorso-lateral parts of each ring. Those (ib., figs. 1–3, *b*, *b*) corresponding with the dorso-lateral horns in *Megalania* are the largest; the laterals, or medilaterals (ib., *c*, *c*) are relatively smaller: the rest of the lateral and underpart of each ring is covered by oblong narrow, flat scales in two transverse series (ib., figs. 2, 3); the anterior (ib., *d*, *d*) short, the posterior series (ib., *e*, *e*) long,—an arrangement which seems to multiply the number of the cutaneous segments. The spines of second degree as to size which spring from the dorsal surface of the tail (ib., fig. 1, *a*, *a*) are six in number on each ring, arranged in the same transverse line: they are shorter than the dorso-lateral series, and slightly diminish toward the mid-line.

All the spines are rather more compressed, more triangular, more sharply pointed, than in *Megalania:* the series *a* and *b*, especially the latter, are relatively larger and longer; but it must be remembered that in the case of the great extinct Lizard we have only the osseous cores of these weapons, and that their sheath of true horny material may have greatly added to the efficiency of the strange and grotesque armature.

In *Uromastix*, moreover, as was noted in Part II. (tom. cit., p. 1047) in reference to the cranial horns of *Moloch*, histological development has not advanced in the diminutive Lizards beyond the fibro-cartilaginous stage of the productions of the corium supporting the corneous sheaths.

Pursuing this track of comparison it was with interest I found in the small existing Australian species that, although the caudal horns or spines are relatively somewhat smaller in *Moloch horridus* than in *Uromastix princeps*, they more closely repeated in number and arrangement the conditions of the caudal armature above described in *Megalania prisca*.

The annular disposition is, in the main, maintained. Taking one of the cutaneous rings near the base of the tail the following horns are present (Plate 66, figs. 4, 5, and 6).

The dorsals (*a*) are in a single pair, as in *Megalania*, with an interspace equal to their own basal breadth. External to these is a dorso-lateral pair (ib., *b*, *b*) divided from the dorsals by a space rather wider than their own basal diameter. Next come a "lateral pair" (ib., fig. 5, *c*) projecting from the middle of the side-surface of each annular series, but placed, in some of them, rather farther back than the dorso-laterals. Then come the spines of a ventro-lateral pair (ib., *d*) nearly on the same vertical line with the laterals; but, as in *Megalania*, of markedly smaller size and projecting from the margin of the under surface of the tail. On this part may be distinguished some still smaller spines, with a scattered arrangement.

At the under surface of the basal part of the tail the horns (*b, c, d*) appear as shown in fig. 6, the surface is flattened and finely tuberculate but without spines. Towards the hinder half of the tail the two horns of the lateral pair deviate from the transverse parallel arrangement, and assume a zig-zag one, but with longitudinal intervals in proportion to their gradually decreasing size: the dorsal spines longest retain their horn-like proportions.

Moloch horridus thus exemplifies a closer resemblance in the number and arrangement of the annular series of caudal horns to *Megalania* than does *Uromastix princeps*, and a closer repetition also in the shape and proportions of the individual horns: they are not compressed, but are more regular cones than in the Zanzibar Lizard. If the apex of the cone be sharper in *Moloch* than in *Megalania*, such difference may be due, as before remarked, to the absence of their corneous covering in the massive bony cores exemplified in the fossils from "King's Creek."

It is probable that the arrangement of these singular caudal weapons in the part of the tail of *Moloch*, which part is wanting in our present evidences of *Megalania*, may indicate the character of the armature of such missing evidences. But that the tail in *Moloch* is different at its terminal portion from the corresponding part in *Megalania* is demonstrable and not any of the horny cones have bony bases.

For a repetition of so singular a structure we must go to a higher class of Vertebrates. Before, however, entering upon comparative details afforded by certain members of the Mammalian class, some further notice may be expected of dermal ossifications, more especially caudal ones, in the class *Reptilia*.

In the group of existing Lizards to which both *Moloch* and *Uromastix* are akin there are genera, *e.g. Scincus, Tiliqua*, in which the scales, or most of them, are supported by bones: in *Cyclodus* the degree of ossification is such as to raise the surface of the skin both of the trunk and tail into prominences, which, however, are obtuse.

The chief dermal ossifications in the order *Chelonia* form, as is well known, the horizontal plates which become confluent with the neural spines and pleurapophyses of the dorsal vertebræ; also with the sternum and sternal ribs: the marginal plates of the carapace are independent skin-bones. But as no such ossifications are associated with the caudal vertebræ, further notice of these *Reptilia* is uncalled for in the present relation. A bony basis of the horny armour of parts of the tegument is constant in the species of the order *Crocodilia*. Along the dorsal and lateral regions, but especially the dorsal, in the neck, trunk, and tail, the outer surface of certain of such scutes is produced into a ridge. In the Alligators ventral as well as dorsal scutes have an osseous support. This character is so remarkable in the extinct *Crocodilia* of the Oolitic period that the Teleosaur of the Caen deposits was characterised by CUVIER as "l'espèce la mieux cuirassée de tout le genre."*

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* 'Ossemens Fossiles,' tom. v., pt. 2, p. 139.

In the Wealden *Goniopholis* the quadrangular dorsal scutes are united, tile-wise, by peg-and-socket joints; the equally thick ventral scutes join each other by thick sutural borders, and an annular arrangement of the armour needs but little to be complete.*

This, in fact, is effected in a more closely mailed Saurian described under the name of *Ætosaurus ferratus*.† Nearly a score of these extinct Reptiles, in all their panoply, were discovered and exposed on one slab of "Middle Keuper sandstone," at Haslach, in Wirtemberg. The arrangement of the scutes is annular. The long axis of the dorsal ones is transverse and a single pair spans the back and corresponding surface of the tail: the ventral scutes are quadrangular, in four transverse pairs: there is a single large lateral scute on each side. Thus twelve scutes compose each dermal segment or circle. But in none is the outer surface produced into a ridge or spine.

In *Hylæosaurus* the petrified osseous supports indicate the back to have been defended by long, compressed sharp-pointed triangular spines, probably in pairs, resembling in shape and relative size the caudal ones of *Uromastix princeps;* but evidence of the dermal caudal armature of the above Dinosaur is wanting.

To this part of *Megalania* the nearest approach in its cold-blooded class seems to be made by a Liassic Dinosaur (*Scelidosaurus Harrisonii*).

The surface of this Saurian was defended by longitudinal series of massive dermal bones, those occupying the median and upper surfaces being arranged in pairs upon the trunk and base of the tail. External to these were lateral series, at least two in number on the trunk, having a similar shape and ridged exterior; but in some of the mid-dorsal series, the bony support of the horny tegument was developed into a stout conical shape, like those on the caudal girdles of *Megalania*. Each endoskeletal segment of the tail of the Scelidosaur was associated with an annular series of detached dermal ossicles of considerable size and thickness. Where best preserved, beyond the base of the tail, they are four in number (Plate 66, fig. 7). The largest (*a*) protects the upper surface of the segment, the next in size (*c*) the under surface, and a pair of smaller scutes (*b*) are applied one on each side of the same segment. These plates are termed "dermo-neural," "dermo-hæmal," and "dermo-lateral" respectively, in the undercited monograph.‡

The dermo-neural plate of the segment figured in Plate 66 is of an oblong shape, with an excavated base, the sides of which converge to an apical elevated ridge, with a longitudinally convex contour. A portion of the left side of the base has been removed to expose the hollow, in which a small depression receives the summit of the neural spine: the fore part of the basal cavity rests upon the upper surface of a produced anterior zygapophysis of the inclosed caudal vertebra.

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* "Report on British Fossil Reptiles," vol. of the British Association, &c., for 1841, p. 70.

† Oscar Fraas, 'Monograph on *Ætosaurus*,' 4to., 1877.

‡ Volume, 4to., issued by the Palæontographical Society, for 1862, p. 22 of the "Monograph on Liassic Reptilia."

The dermo-hæmal ossification (ib., ib., *c*) resembles the dermo-neural one (*a*) in a reversed position; its base is also excavated, and receives in a special depression the end of the hæmal spine; it is not situated exactly opposite the one above but is further back. In both bones the outer surface is finely rugose, the inner or concave surface is smooth.

The dermo-lateral ossicles are oblong and sub-elliptic; similarly rugose externally, smooth in the inner surface, and not directly attached or applied to any part of the corresponding vertebra. In the segment selected for Plate 66, fig. 7—about the 20th of the caudal series—the transverse processes of the vertebra have ceased to be developed.

It is obvious that if these detached portions of the dermo-skeleton of a caudal segment of the Scelidosaur had been connected by continuous ossification in the intervening corium, a bony cylinder would have resulted, such as the extinct *Megalania* presents: but with the difference as regards the number of the outwardly projecting processes.

The closer repetition of the Megalanian caudal character is presented by members of the Mammalian class. In the gigantic extinct species of the loricate family of the order *Bruta* forming the type of the genus *Glyptodon*,* the extent of tail preserved measured in the specimen on which the genus was founded 3 feet 6 inches in length, and was inclosed in a defensive osseous sheath, showing an annular arrangement of the skin-bones.

In the basal half of this sheath the constituent ossicles are arranged in segments, and each segment is composed of two annular series of ossicles, the proximal series presenting the rose-pattern of those of the main part of the carapace, the distal series of larger and more prominent ossicles repeating rather the character of the hinder marginal series of the carapacial bones.† The constituent ossicles have more or less coalesced in each segment; and, of these, the seven basal rings remain distinct, allowing a certain extent of flexibility in that part of the tail. The terminal portion of the tail-sheath forms, as in *Megalania*, one continuous osseous case, closed at the tip, near which it developes two lateral large and massive conical bosses, the apex being obtuse. A series of progressively smaller and less prominent. lateral ones indicates six or more rings to have contributed to this bony sheath; and smaller supplementary elliptical plates, above and below the larger lateral ones, complete the complex defensive peripheral developments of the osseous tail-armour of *Glyptodon clavipes.* But the species of *Glyptodon* of which the tail-sheath more closely resembles that of *Megalania*, as exemplified by the subjects of the present paper, is the *Glyptodon* (*Schistopleurum*) *asper* of BURMEISTER.*

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* 'Appendix' to PARISH (Sir WOODBINE) 'Buenos Ayres from the Conquest,' 8vo., 1838, pp. 217 and 433, plate 1; Proceedings of the Geological Society of London, March, 1839, p. 236; Trans. Zool. Soc. (2nd series), vol. vi., p. 81.

† The figure of the carapace in my 'Catalogue of the Fossil Mammals in the Museum of the Royal College of Surgeons of England' is copied by NODOT ('Description d'un nouveau genre d'Edenté fossile,' plate 4): a more complete restoration of the entire exo-skeleton and exposed parts of the endo-skeleton forms the subject of plate 36 of the excellent work of BURMEISTER (German M. and Phil. D., Director of the Public Museum of Buenos Ayres), entitled 'Anales del Museo Publico de Buenos Ayres,' 4to., 1870–1874.

The bony tail-rings are nine in number (Plate 66, fig. 8), and each ring is composed, as in *Glyptodon clavipes*, and, partially, in *Uromastix princeps*, of a smaller and larger series of ossicles (ib., *d*, *e*); but those of the hinder series (*e*) are each developed into a conical protuberance, smallest in the two proximal rings and increasing in size, prominence, and sharpness of apex, to the fourth ring; thence the horn-like bosses decrease in number, though hardly in size, to the terminal closed segment of the sheath.

In an antecedent girdle may be distinguished one dorsal pair (Plate 65, fig. 5, *a*, *a*), a dorso-lateral pair (ib., *b*, *b*), a ventro-lateral pair (ib., *c*, *c*), and two smaller ventral pairs (*d*, *d'*), of massive conical protuberances.

In number and arrangement the pattern of the small existing Lacertian (*Moloch*) is adhered to; but the great Armadillo, in the osseous basis of the protuberances as well as in their relative size, repeats more closely the armature of the terminal segments of the bony tail-sheath of *Megalania.* The relations of the exo- to the endo-skeleton are also essentially the same. The neural spine of the caudal vetebra, in *Glyptodon*, rises to contact with the roof of the overarching bone, but expands and divides to give a more extended support (ib., ib.). The hæmal spine in like manner expands, but in a minor degree, upon the floor of the cylinder. Moreover, in the caudal segment, the subject of fig. 5, the transverse processes are developed and similarly expand to buttress the side walls of the dermosteal sheath.

In the caudal segment of *Megalania*, the subject of Plate 64, both neural and hæmal spines do expand at their terminal confluence with the sheath, and if, as is probable, the lack of extension of the transverse processes (*d*, *d*) be due to accidental fracture and loss, the inward projections of the parts of the sheath against which, if entire, they abutted, indicate, also, expanded terminations. But the difference in the relative size of the endoskeletal element of the tail-segment is manifestly in favour of the Mammal.

The unlooked for anticipation, so to speak, of the loricate character of tail of certain members of placental Mammals by a genus of oviparous Lacertians has suggested the following final remarks.

The Pangolins (*Manis*) offer a singular exception in their high class to other Mammalia in the scaly nature of their tegument in which the imbricate arrangement of the horny tissue repeats the common condition of the covering of Lizards. I conceive this, with their toothless character, their gizzard and gastric glands, and their Reptilian retention of intra-abdominal testes, to contribute a more significant indication of their position in the scale of the warm-blooded, viviparous, and lactiferous

* 'Anales del Museo Publico de Buenos Ayres,' 4to., 1871, p. 411, plate 40.

class than the shape or substance of any portion of the deciduous appendages of their fœtus.

The dermal bones of the Armadillos (*Dasypus*) have been cited, in like relation to guidance as to status in the Mammalian series; and now, for the first time, we are enabled to show that the singular annular arrangement of such ossifications, sustaining massive conical cores of corneous weapons, is a repetition of, or, as it were, has been borrowed from, an antecedent cold-blooded, air-breathing class.

And what are the Mammals that most resemble the scale-clad and bone-clad genera in other organic phenomena? I refer not, now, to the Monotremes and Marsupials; these, by common consent of Systematists, are deposed from the place they hold in the Cuvierian system, and are relegated to the lowest position in their class. They manifest marked differences amongst themselves, but there is an anatomical character common and peculiar to them. It is a cerebral one, in which the great commissure of the brain, as it exists in all other and higher Mammals, is represented by a rudiment, the connexions and position of which are best expressed by the term "hippocampal commissure."

With the Pangolins and Armadillos are associated, also by a cerebral character, the Rodents, Insectivores, Bats, and Sloths. In every species of these Orders, as in the species of *Bruta*, the cerebral hemispheres are not extended over any part of the cerebellum, and, with few exceptions, are outwardly smooth or unconvolute. At such a stage of Mammalian development is it that we find, as in the Lizards and Snakes, species becoming torpid in winter, of which Dormice among the *Rodentia* and Bats among insect-eaters are notable instances. In such low Mammals the heart, as in *Reptilia*, retains the faculty of circulating carbonised or black blood. Again, as the Pangolin borrows the scales and the Armadillo the scutes of the Reptiles, so the Hedgehogs and Porcupines borrow, for their covering, the main part of the feather, that, namely, to which it is reduced in the wings of the Cassowary.

The supernumerary neck-vertebræ and their free or floating ribs is an osteological character which has been continued from the Crocodiles to the Sloths; and the latter smooth-brained Mammals show kinship with the antecedent cold-blooded air-breathing Vertebrates, by the tenaceous post-mortem irritability of the muscular fibre and the long continuance of its contractile response to stimulants.

DESCRIPTION OF THE PLATES.

PLATE 64.

Front view of the ante-penultimate caudal segment, showing the exo- and endo-skeletal parts; natural size: *Megalania prisca.*

PLATE 65.

Fig. 1. Side view of the four terminal exo-skeletal segments of the tail: ***Megalania prisca.***

Fig. 2. Upper view of the same segments.

Fig. 3. Under view of the same segments.

Fig. 4. Back view of the foremost segment. (The above figures are reduced in the degree shown by comparison with Plate 64.)

Fig. 5. Front view of corresponding caudal segment, showing the exo- and endo-skeletal parts; half the natural size. *Glyptodon asper* (Burmeister).

PLATE 66.

Fig. 1. *Uromastix princeps;* natural size (after O'Shaughnessy*).

Fig. 2. Side view of exo-skeleton of the tail of ditto; natural size (original).

Fig. 3. Under view of the same part (original).

Fig. 4. Upper view of two exo-skeletal segments of the tail; natural size. ***Moloch horridus.***

Fig. 5. Side view of three exo-skeletal caudal segments; natural size. Ib.

Fig. 6. Under view of three exo-skeletal segments, nearer the end of the tail; natural size. Ib.

Fig. 7. Side view of the endo- and exo-skeletal parts of a caudal segment; natural size. *Scelidosaurus Harrisonii.*

Fig. 8. Side view of the exo-skeletal segments of the tail; one-eighth the natural size. *Glyptodon asper* (after Burmeister).

Fig. 9. Upper view of the five terminal exo-skeletal segments of the tail; one-eighth the natural size. *Glyptodon asper* (after Burmeister).

Fig. 10. Under view of the same segments (after Burmeister).

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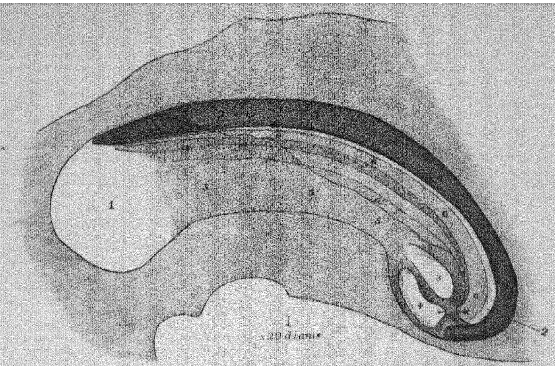

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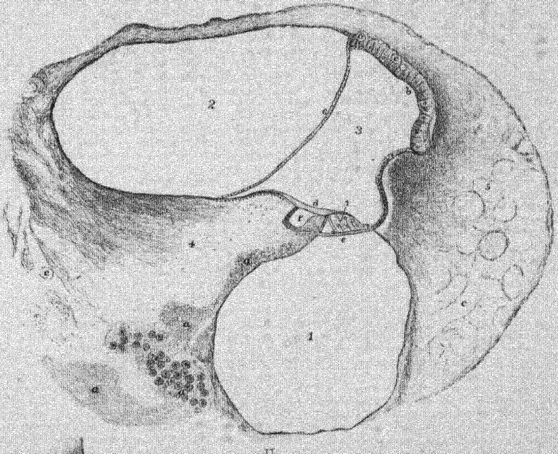

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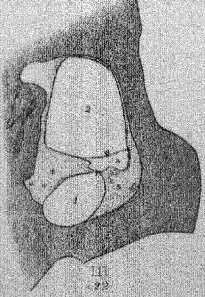

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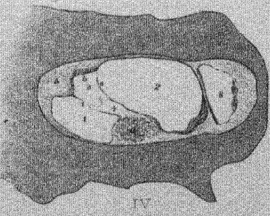

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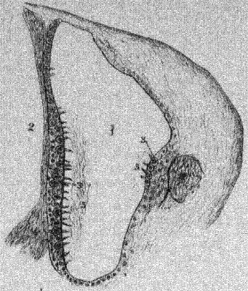

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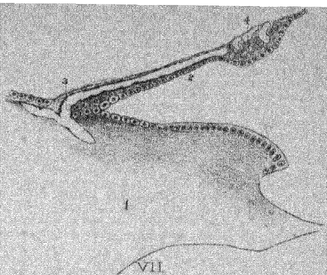

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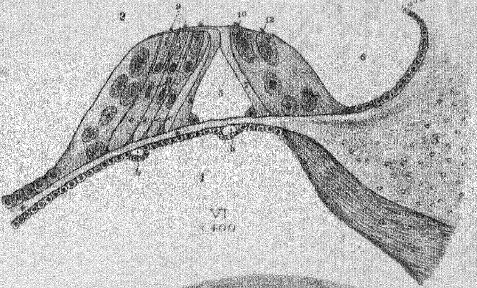

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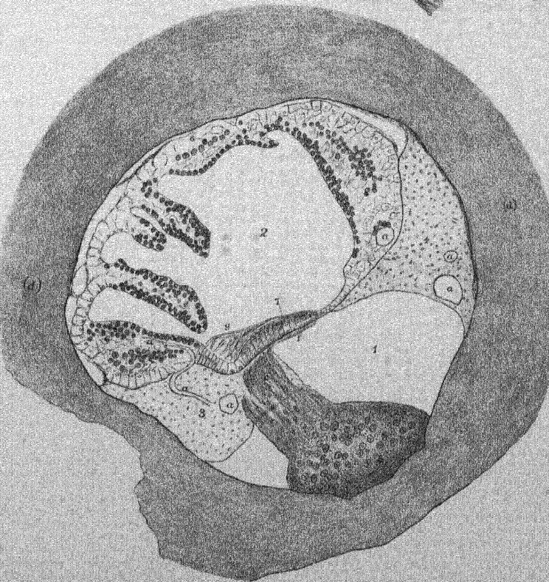

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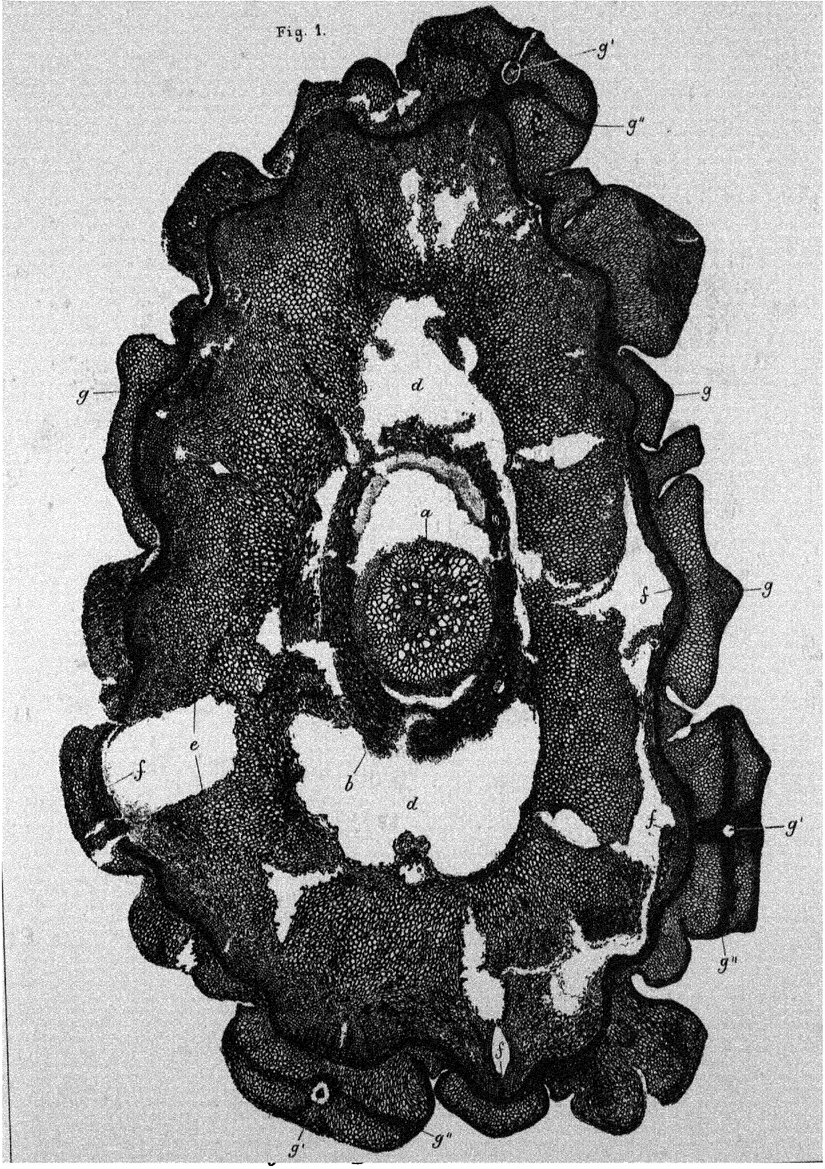

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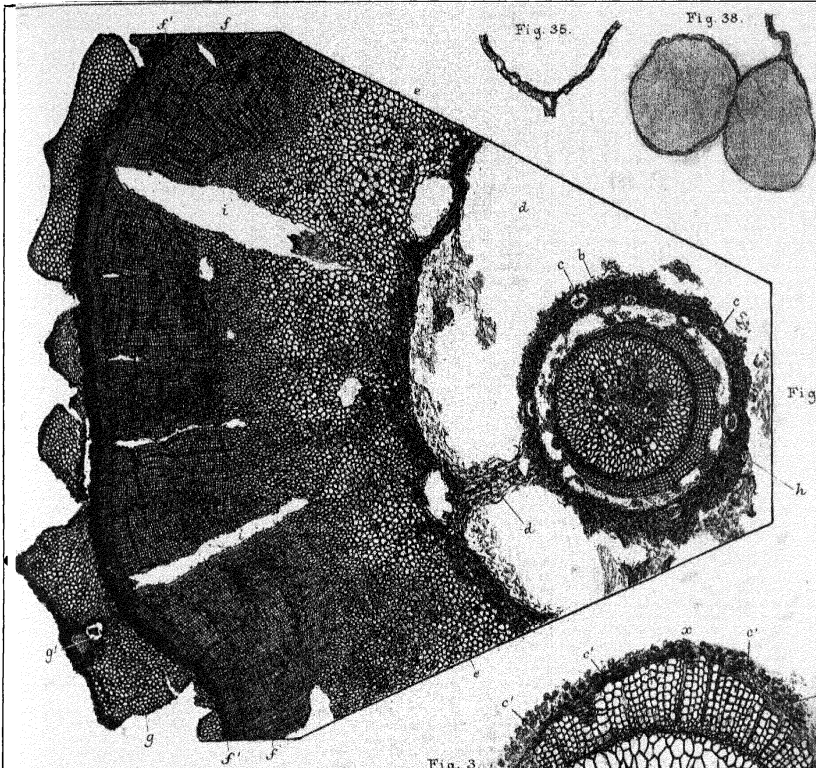

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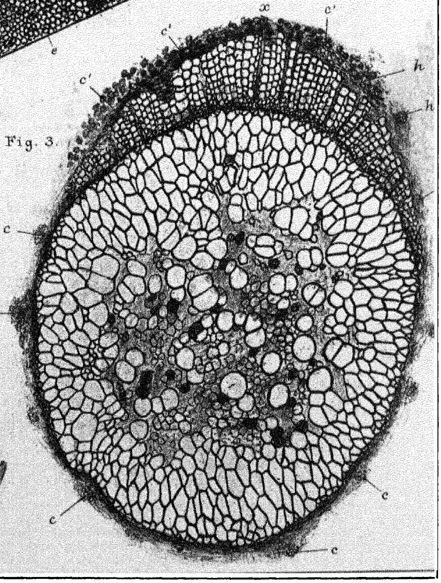

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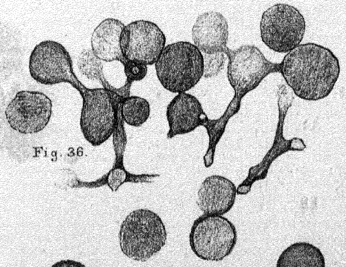

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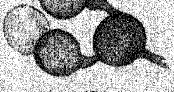

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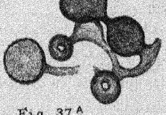

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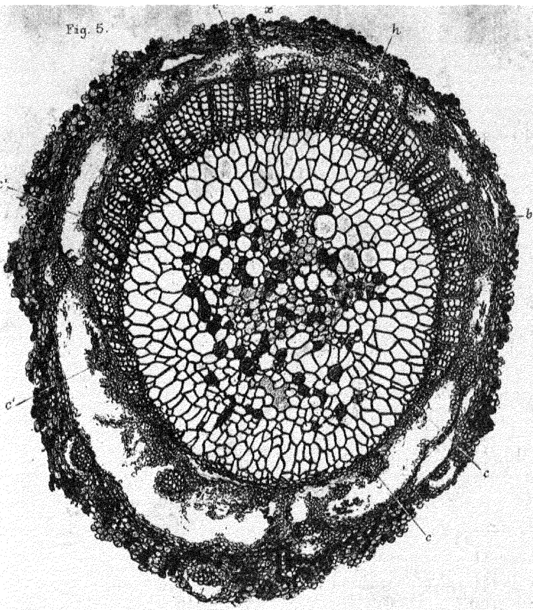

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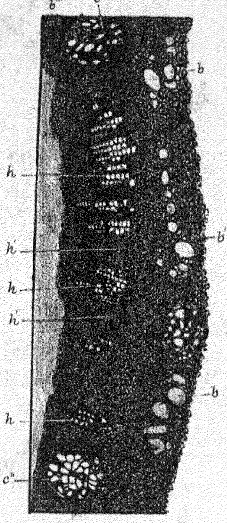

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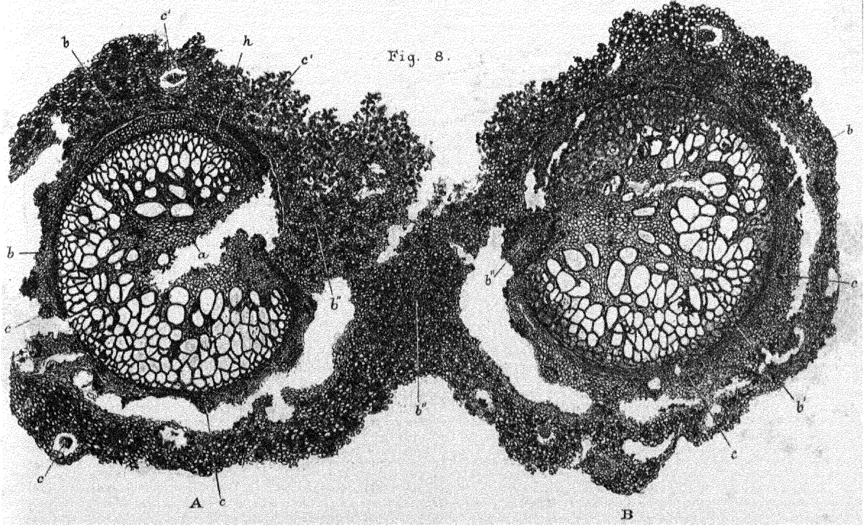

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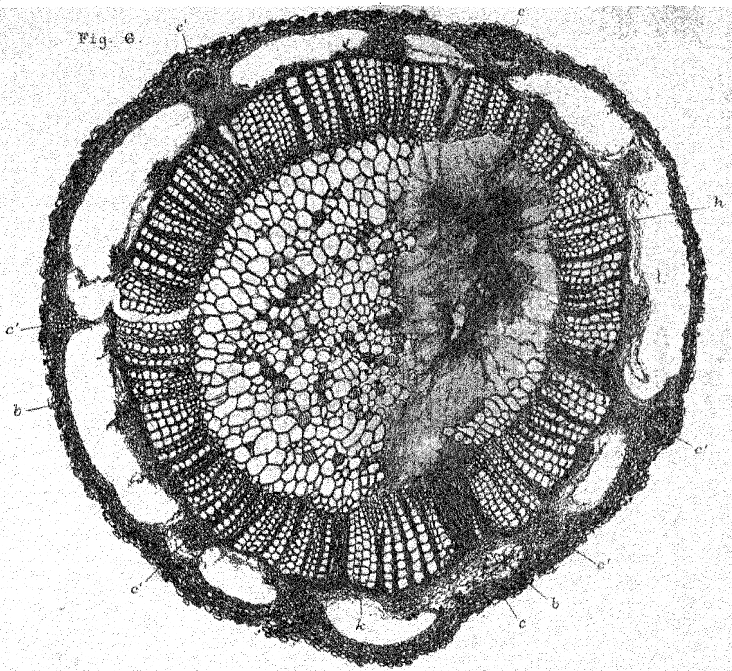

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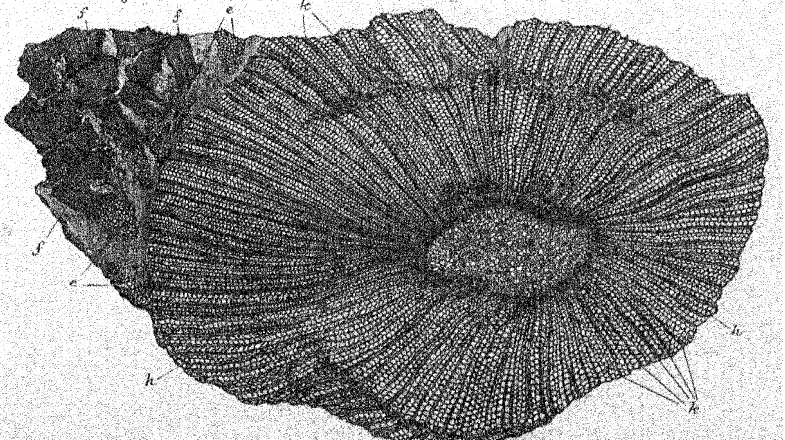

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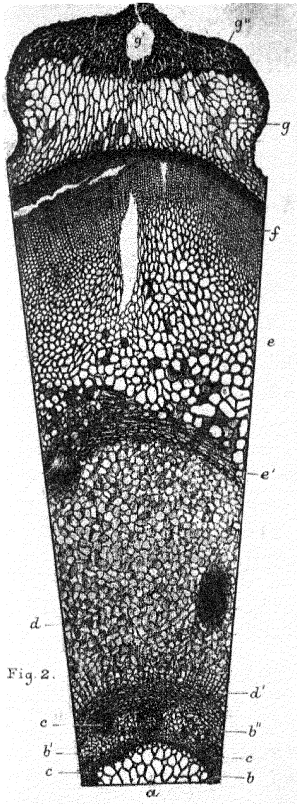

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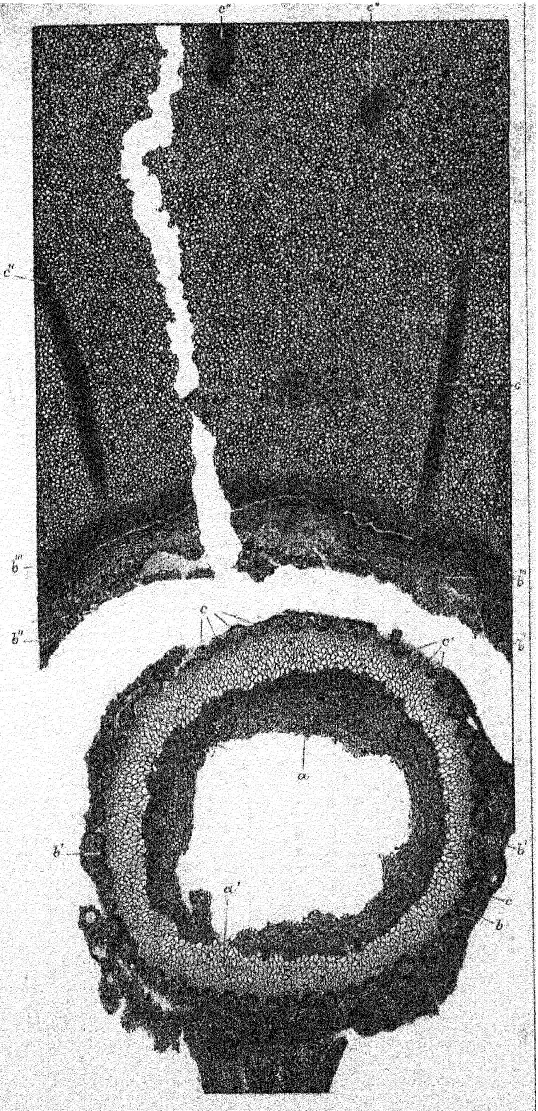

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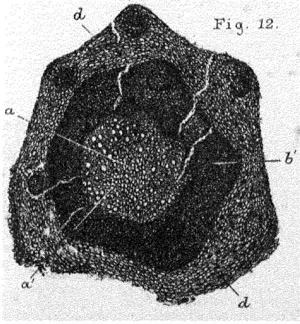

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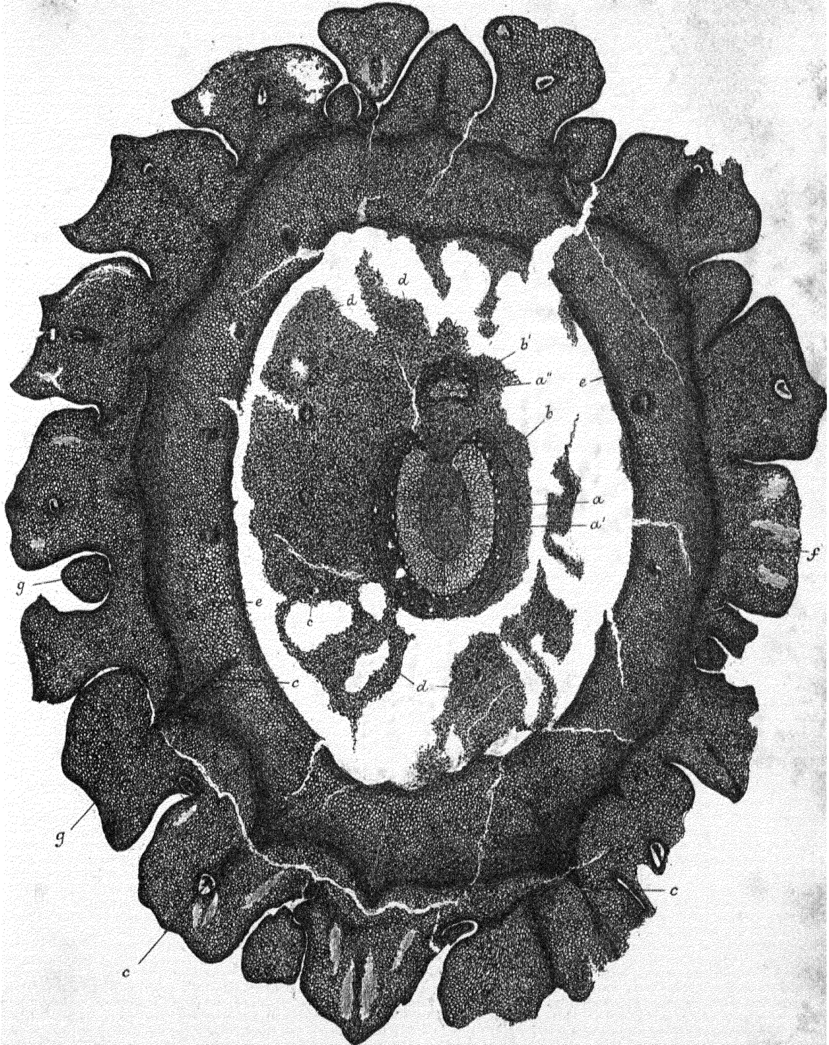

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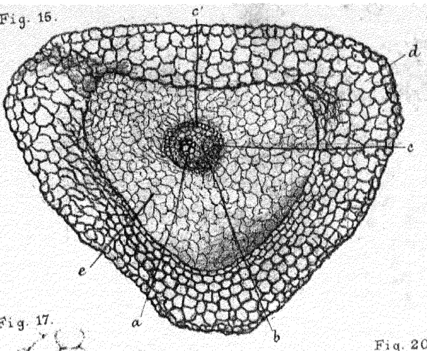

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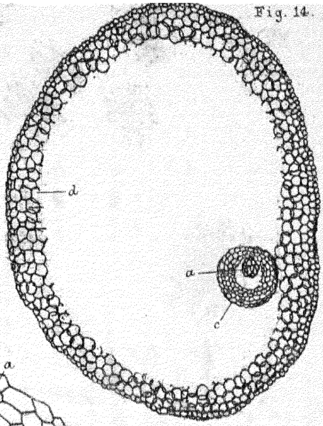

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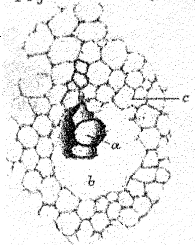

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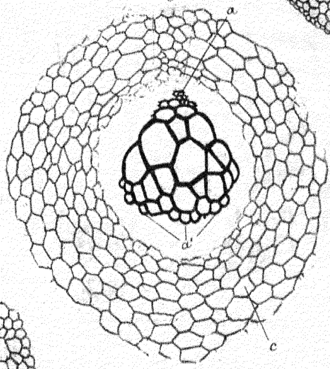

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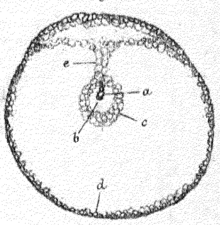

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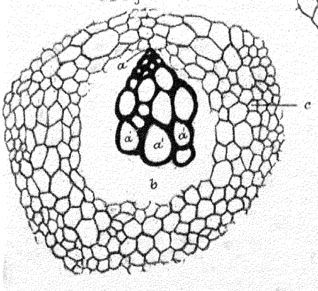

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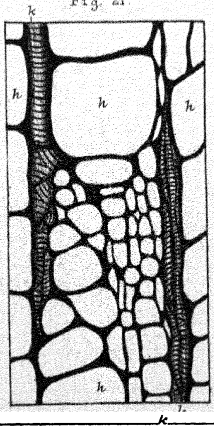

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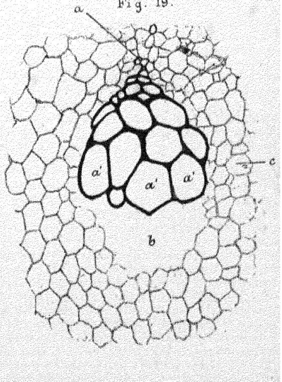

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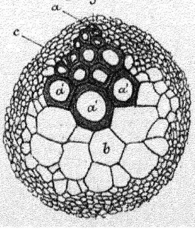

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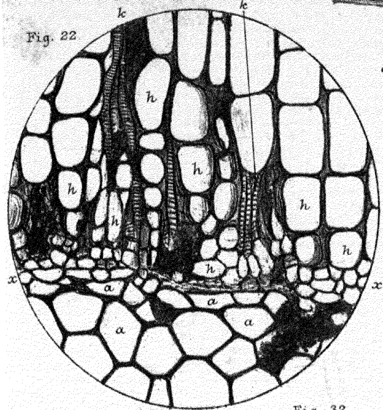
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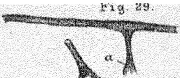

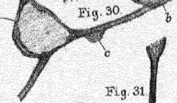
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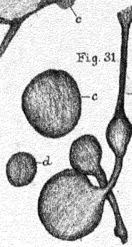
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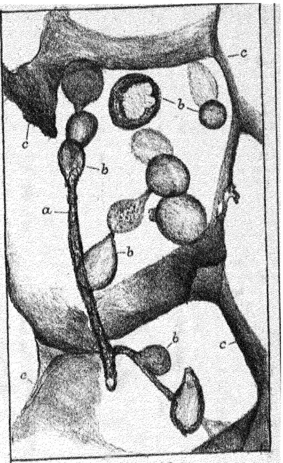
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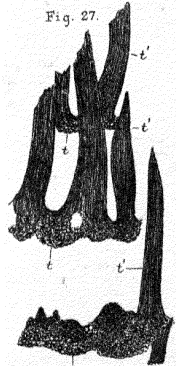
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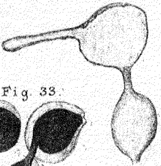
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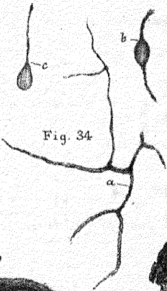
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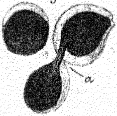
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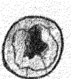
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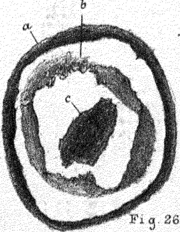
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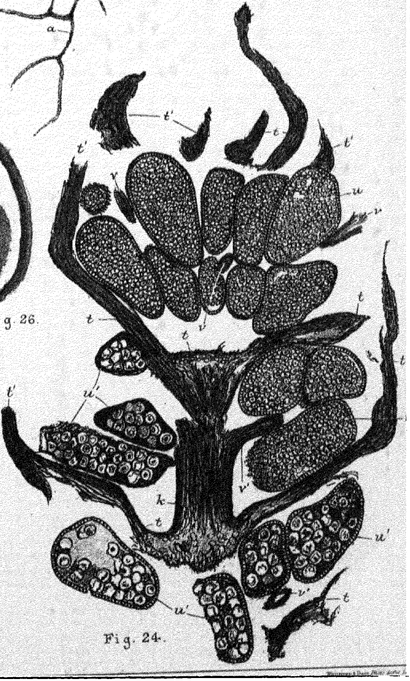
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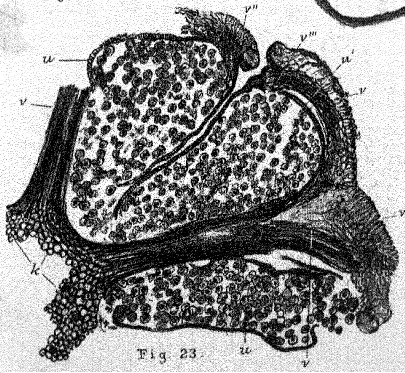
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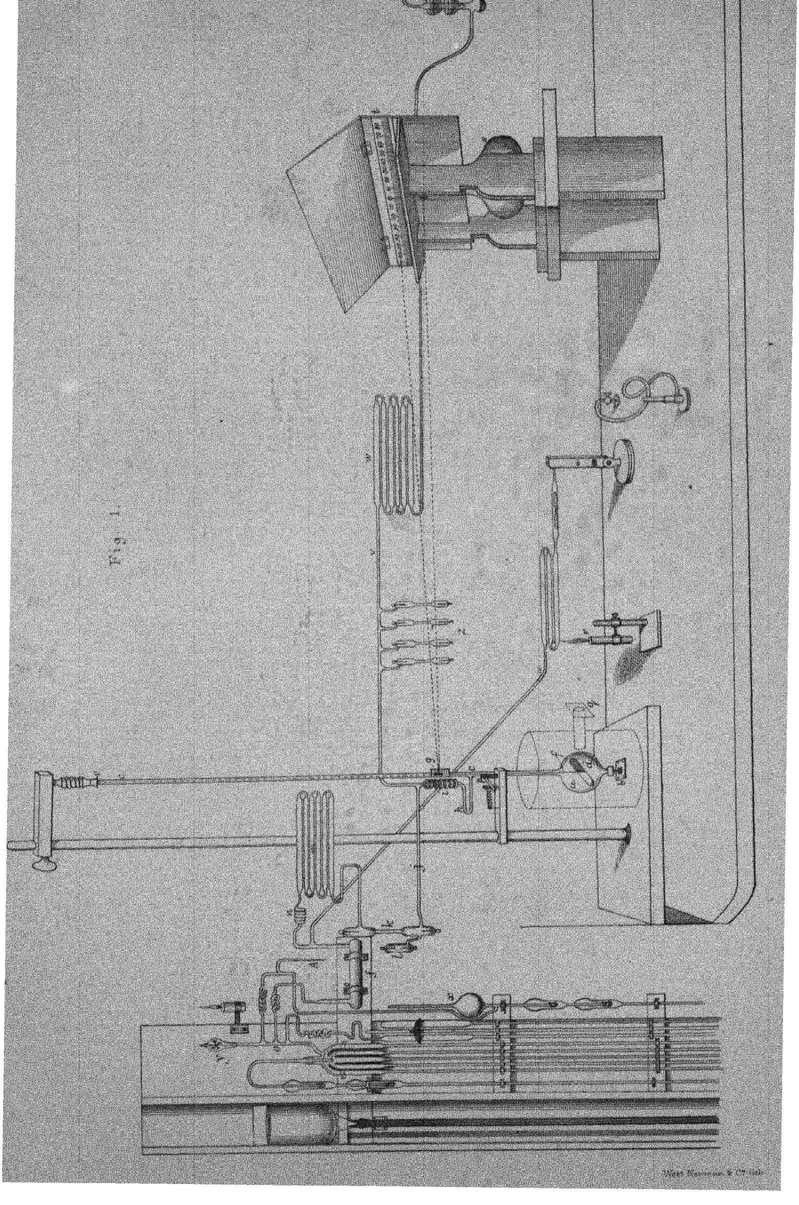

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Phil. Trans. 1881. *Plate* 56.

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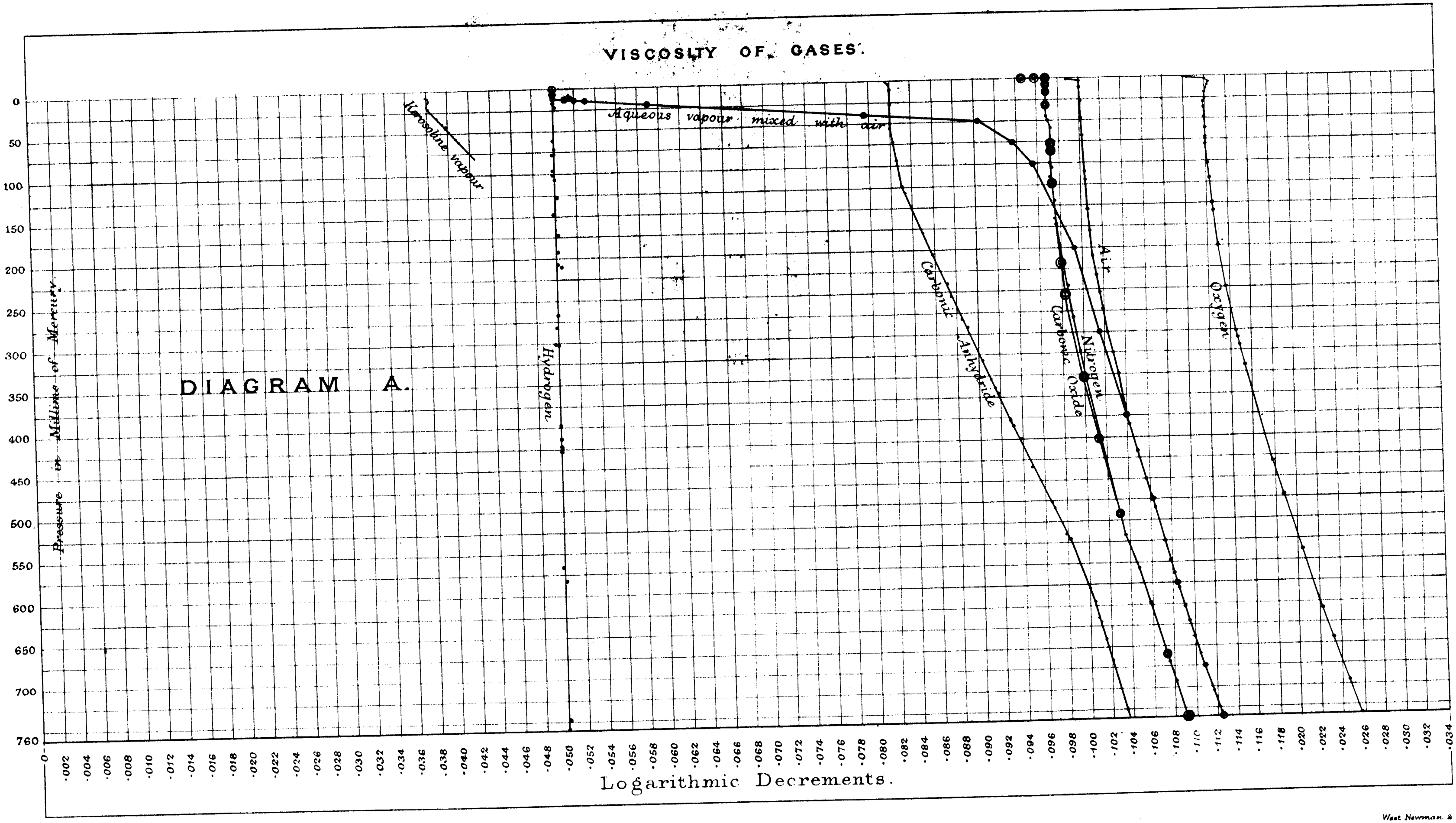

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